Combinatorics of Pisot substitutions

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Sous la direction de
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Plan

- Substitutions: introduction

I. Pisot substitutions

- Dynamics of Pisot substitutions
- Combinatorial tools: dual substitutions
- Applications
- Topology: Rauzy fractals with countable fundamental group

II. Stepping back from the Pisot case, (un)decidability

- Combinatorial substitutions
- Affine iterated function systems

Substitutions: introduction

Example: the Prouhet-Thue-Morse substitution:

- $\sigma : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 21 \end{cases}$

Substitutions: introduction

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Substitutions: introduction

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Substitutions: introduction

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Substitutions: introduction

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Many uses and properties:

- Number theory [Prouhet 1851]
- Combinatorics on words [Thue 1912]
- Dynamics [Morse 1920s]
- Chess! [Euwe 1929]
- $\ldots$

Substitutions: introduction

$$1 \mapsto \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}$$

$$2 \mapsto \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}$$

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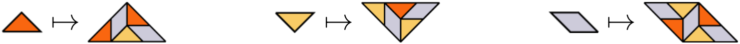
Substitutions: introduction



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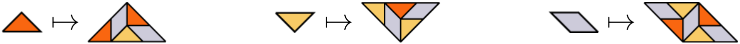

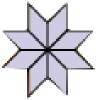

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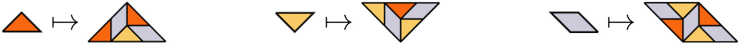

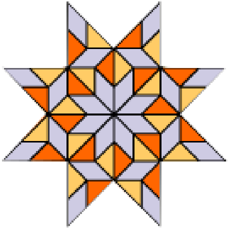

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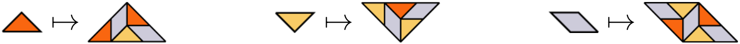

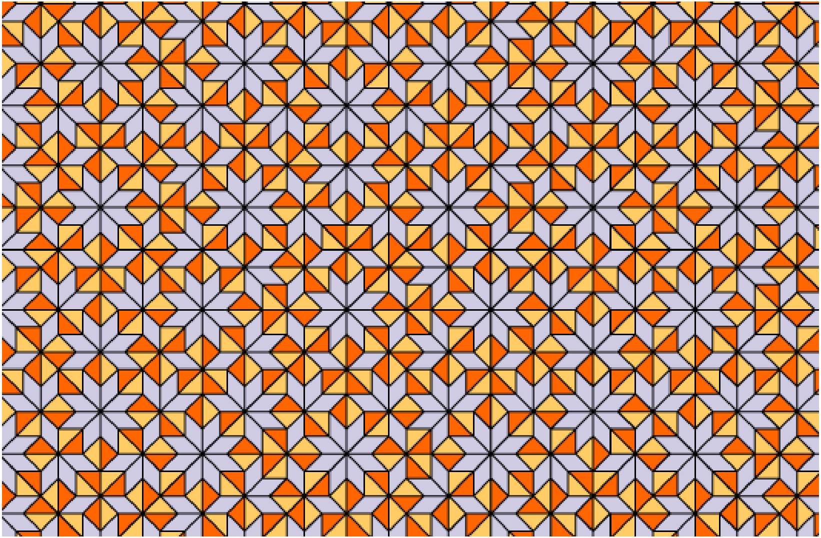

Substitution dynamical systems

Example: the Fibonacci substitution

- $\sigma\ :\ 1 \mapsto 12,\ 2 \mapsto 1$
- $x = 121121211211212112121121121121211211212112121121\ldots$
- Symbolic dynamical system (X_σ, shift)

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  - $\text{shift}(x)_n = x_{n+1}$
  - $X_\sigma \ = \ \text{closure}(\{\text{shift}^n(x) : n \in \mathbb{Z}\}) \ \subseteq \ \{1,2\}^{\mathbb{Z}}$

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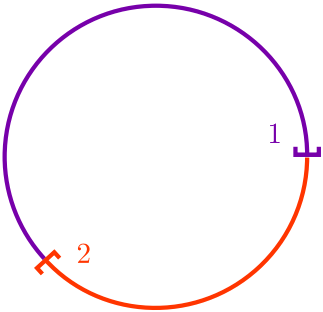

Coding of the orbit of 0.25 in $[0,1]$:

Substitution dynamical systems

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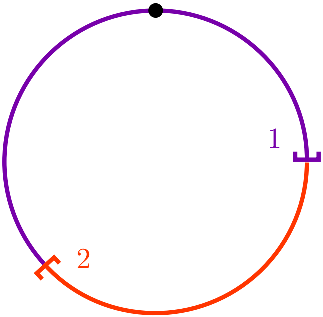

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Substitution dynamical systems

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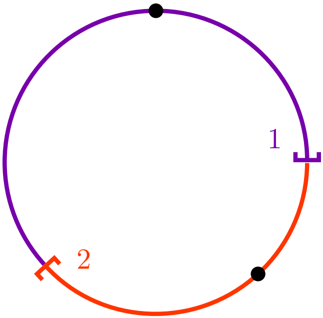

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12

Substitution dynamical systems

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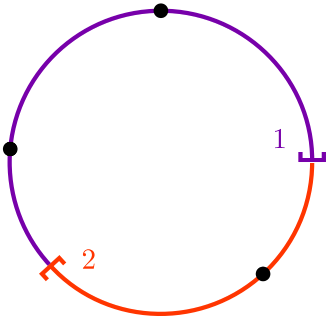

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121

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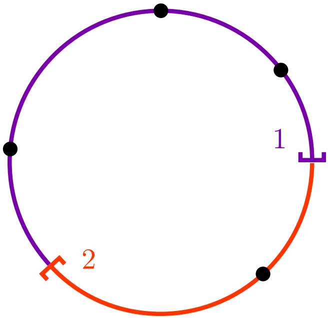

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1211

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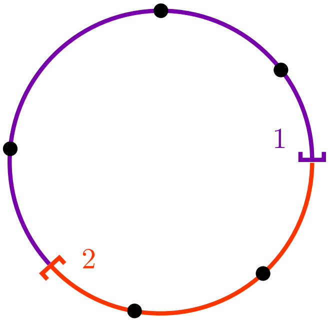

Coding of the orbit of 0.25 in $[0,1]$:
12112

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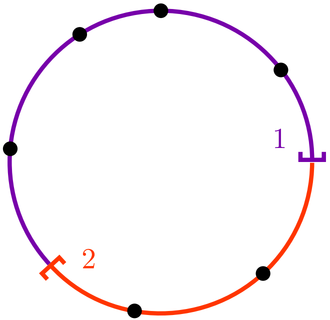

Coding of the orbit of 0.25 in $[0,1]$:
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Substitution dynamical systems

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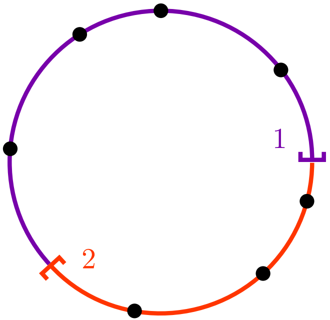

Coding of the orbit of 0.25 in $[0,1]$:
1211212

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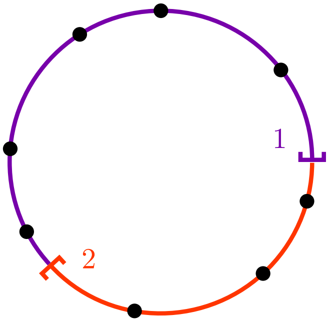

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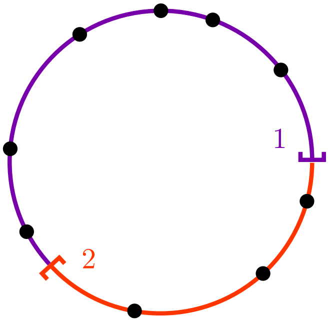

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Substitution dynamical systems

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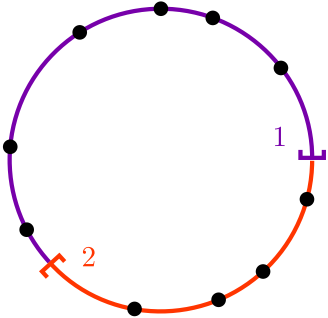

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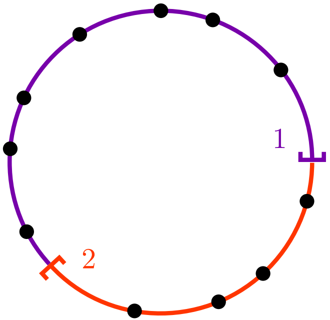

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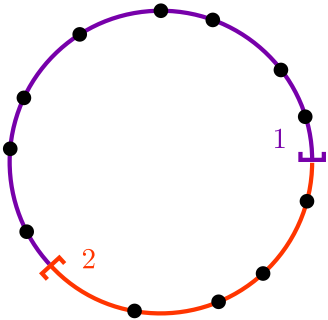

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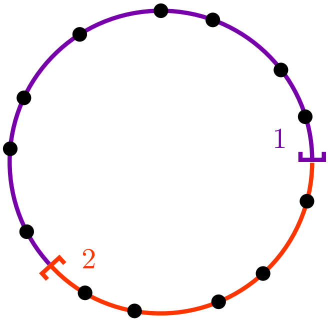

Coding of the orbit of 0.25 in $[0,1]$:
$1211212112112\ldots \in X_\sigma$

Substitution dynamical systems

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- What about the general case?

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Example: $\sigma : 1 \mapsto 12, 2 \mapsto 11$

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Example: the Tribonacci substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

- $(X_\sigma, \text{shift}) \cong (\mathbb{T}^2, x \mapsto x + (\frac{1}{\beta}, \frac{1}{\beta^2}))$
  with $\beta \approx 1.839$, the dominant root of $x^3 - x^2 - x - 1$

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➥ β is a **Pisot number**

Plan

- Substitutions: introduction

I. Pisot substitutions

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I. Pisot substitutions

- **Dynamics of Pisot substitutions**

Pisot substitutions

$$\sigma : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 13 \\ 3 \mapsto 1 \end{cases} \qquad \mathbf{M}_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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Dominant eigenvalue β is a Pisot number:

- Real eigenvalue $\beta > 1$ of $\mathbf{M}_\sigma$ $\qquad \beta \approx 1.839$
- Conjugates $|\beta'|, |\beta''| \in]0, 1[$ $\qquad \beta', \beta'' \approx -0.419 \pm 0.606i$

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Action of M_σ on $\mathbb{R}^3$:

- **Expanding line $\mathbb{E}$**
- **Contracting plane $\mathbb{P}$**

Rauzy fractals

$$\sigma^\infty(1) = 121312112131212131211\cdots$$

$\pi : \mathbb{R}^3 \to \mathbb{P}$,
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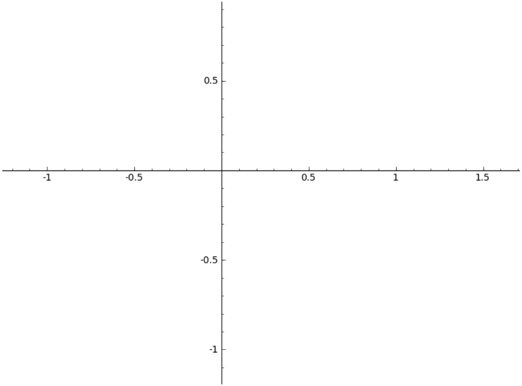

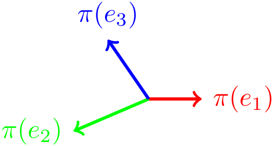

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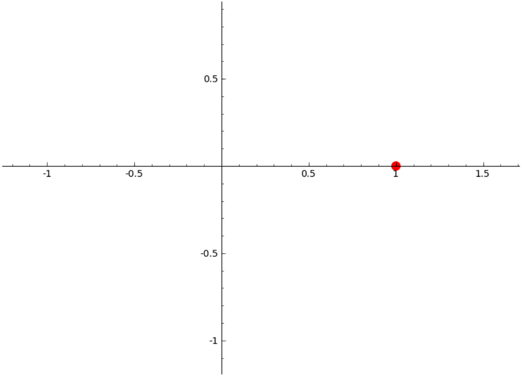

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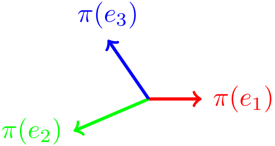

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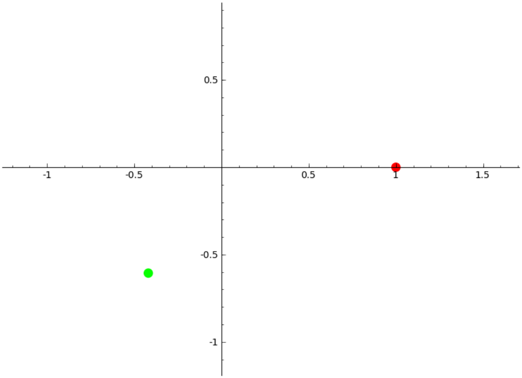
$\pi(e_1 + e_2)$



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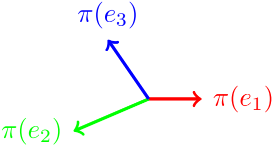

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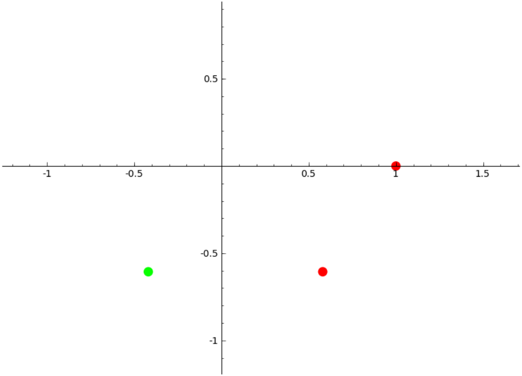

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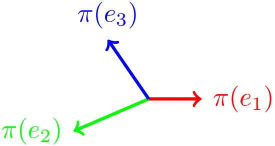

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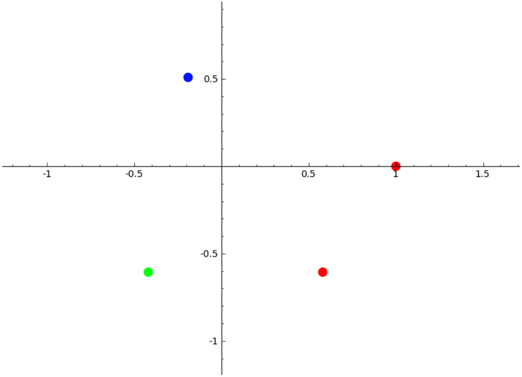

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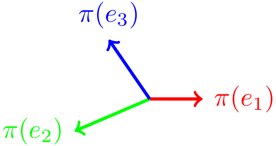

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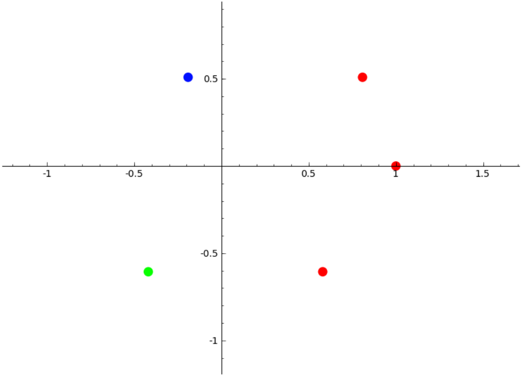

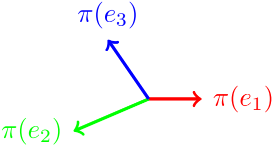

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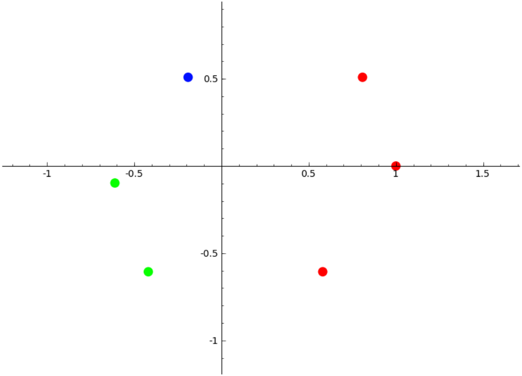

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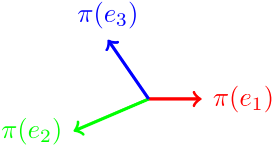

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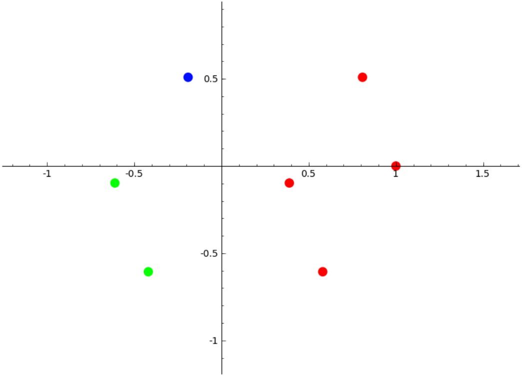

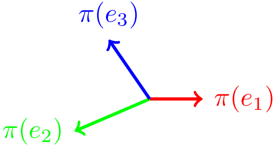

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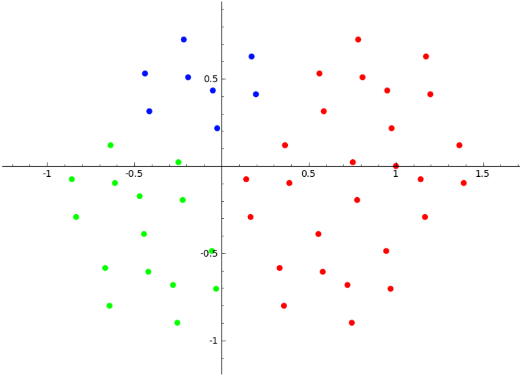

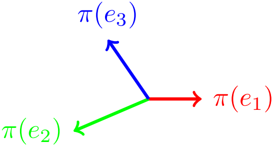

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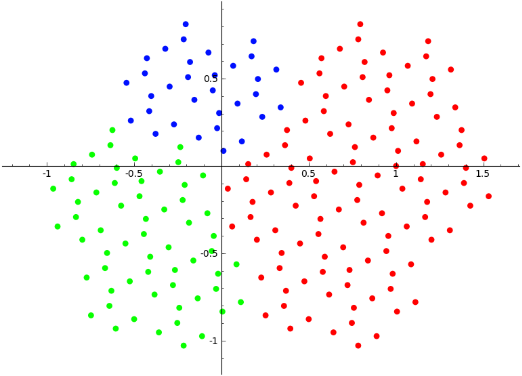

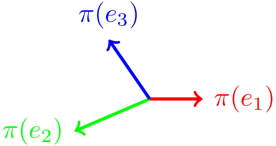

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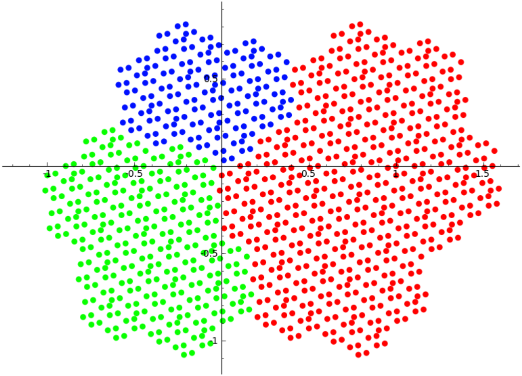

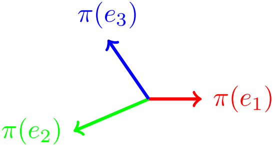

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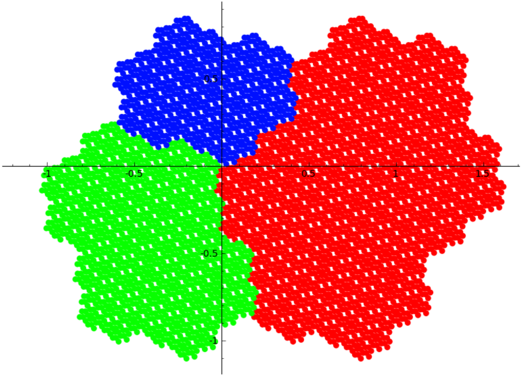

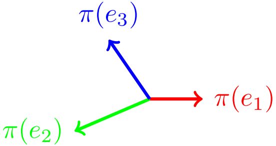

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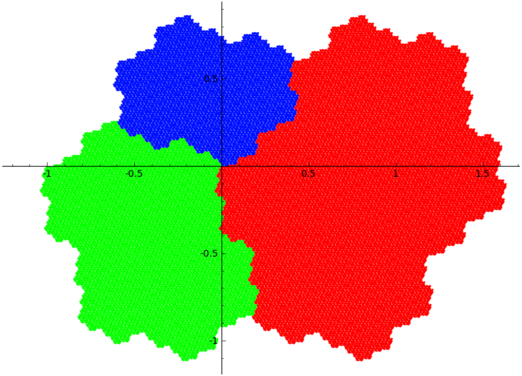
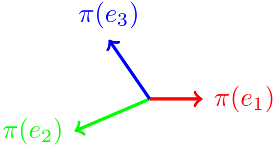
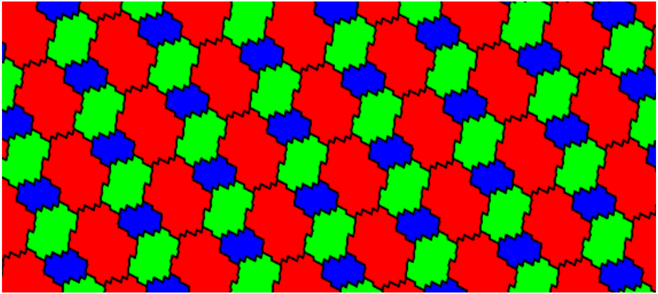




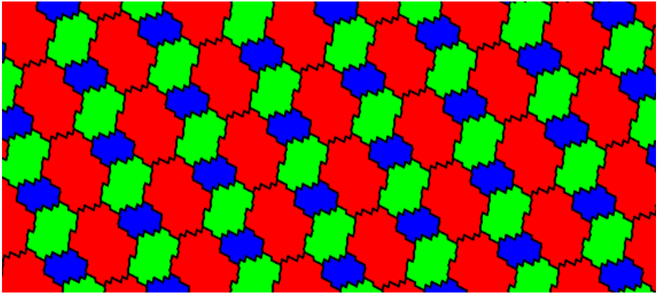
Dynamics of Pisot substitutions

Periodic tiling



Dynamics of Pisot substitutions

Periodic tiling $\longleftrightarrow$ partition of the 2D torus $\mathbb{T}^2$



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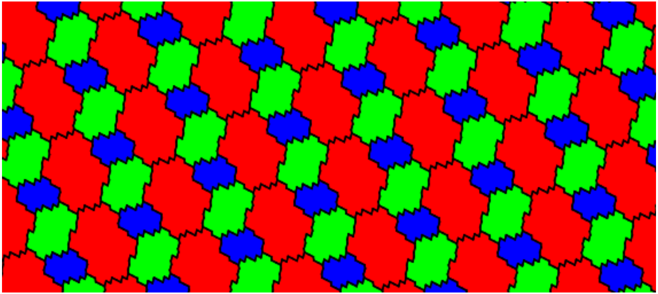

Allows to prove $(X_\sigma, \text{shift}) \cong (\mathbb{T}^2, x \mapsto x + (\frac{1}{\beta}, \frac{1}{\beta^2}))$

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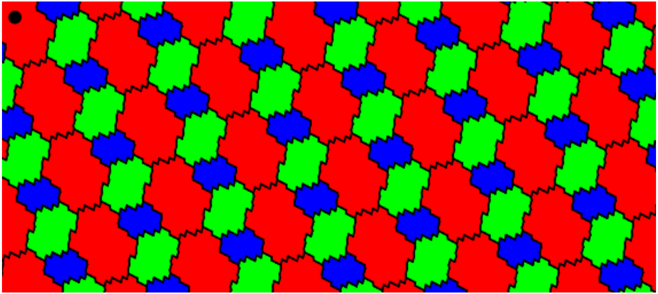

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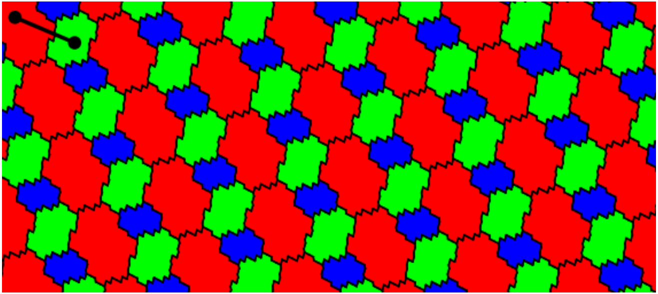

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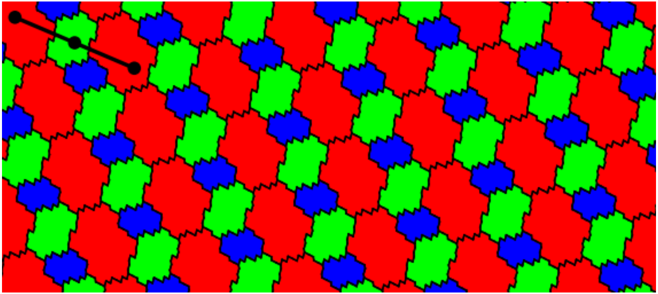

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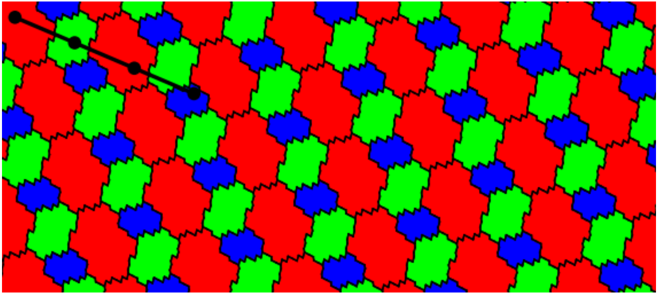

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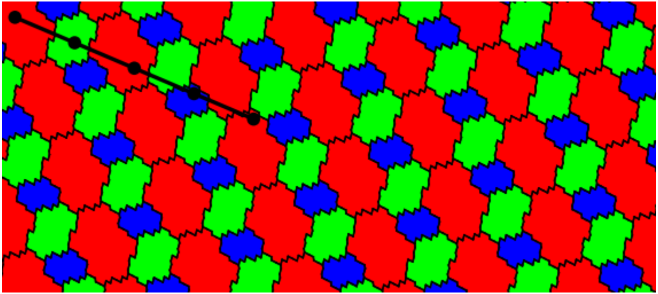

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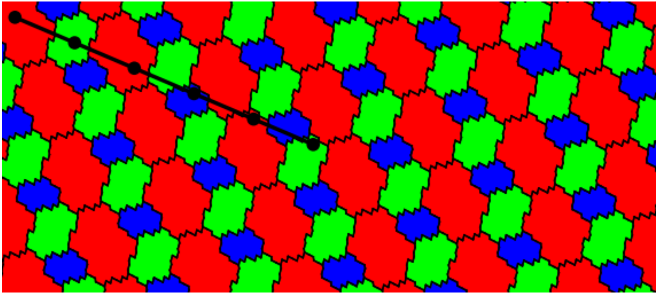

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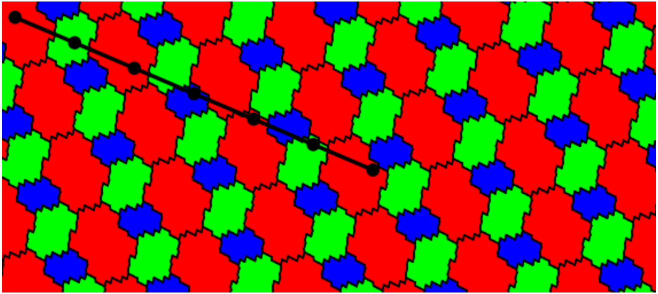

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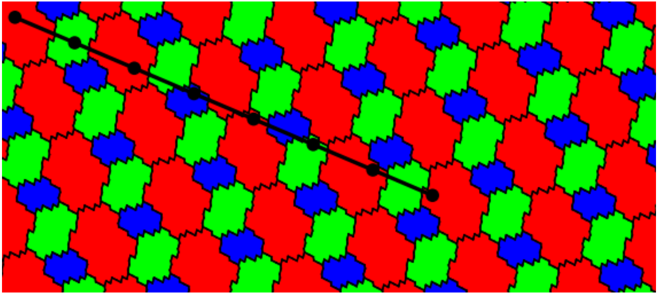

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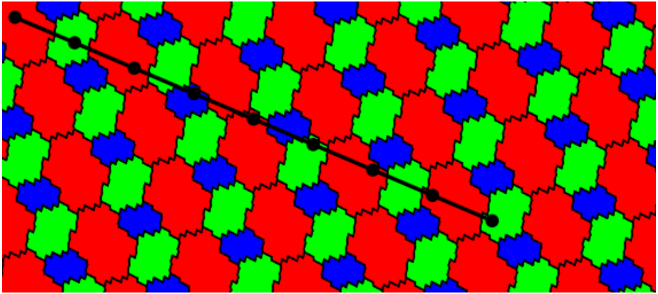

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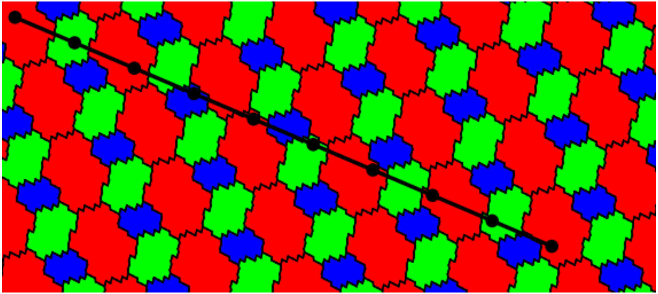

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For every unimodular primitive Pisot σ:

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Combinatorics	Coincidence	(Dekking, Livshits)
	Balanced pairs	(Livshits, Sirvent & Solomyak)
Arithmetics	Finiteness property	(Solomyak)
	Homoclinic condition	(Vershik, Schmidt)
Tiling	Strong overlaps	(Solomyak)
Physics	Algebraic	(Lee)
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- 2-letter case proved [Hollander-Solomyak, Barge-Diamond]

Products of Pisot substitutions

Brun substitutions

$\sigma_1 : 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 32$

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Why Brun substitution?

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- **Dynamics:** tackle infinite families
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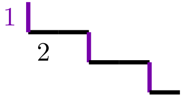
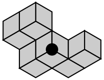
  - S-adic systems
- Generalize what is known for **Sturmian sequences:**

	2 **letters**	3 **letters**
Algorithm:	Euclid (cont. frac.)	Brun
Substitutions:	$1 \mapsto 1,\ 2 \mapsto 21$ $1 \mapsto 12,\ 2 \mapsto 2$	$1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 32$ $1 \mapsto 1,\ 2 \mapsto 3,\ 3 \mapsto 23$ $1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 13$
Geometric:	 discrete lines	 discrete planes

Plan

- Substitutions: introduction

I. Pisot substitutions

- Dynamics of Pisot substitutions
- **Combinatorial tools: dual substitutions**

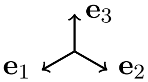
Combinatorial tools: unit faces

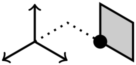
Unit face $[\mathbf{x}, i]^*$, of type $i \in \{1, 2, 3\}$ at $\mathbf{x} \in \mathbb{Z}^3$

$$[\mathbf{x}, 1]^* = \{\mathbf{x} + \lambda \mathbf{e}_2 + \mu \mathbf{e}_3 : \lambda, \mu \in [0, 1]\} =$$ 

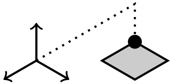
$$[\mathbf{x}, 2]^* = \{\mathbf{x} + \lambda \mathbf{e}_1 + \mu \mathbf{e}_3 : \lambda, \mu \in [0, 1]\} =$$ 

$$[\mathbf{x}, 3]^* = \{\mathbf{x} + \lambda \mathbf{e}_1 + \mu \mathbf{e}_2 : \lambda, \mu \in [0, 1]\} =$$ 





$[(-1, 1, 0), 1]^*$



$[(-3, 0, -1), 3]^*$

Combinatorial tools: unit faces

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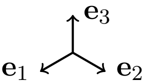

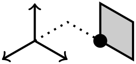

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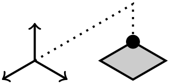
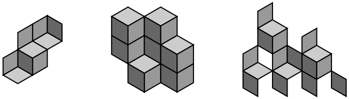


$[(-3, 0, -1), 3]^*$

"Two-dimensional words": collection of faces:



Combinatorial tools: $\mathrm{E}_1^*(\sigma)$ dual substitutions

Definition [Arnoux-Ito 2001]

Let $\sigma : \{1,2,3\}^\star \to \{1,2,3\}^\star$ such that $\det(\mathbf{M}_\sigma) = \pm 1$.

$$\mathbf{E}_1^*(\sigma)([\mathbf{x}, i]^\star) \; = \; \mathbf{M}_\sigma^{-1}\mathbf{x} + \bigcup_{(p,j,s)\in\mathcal{A}^\star\times\mathcal{A}\times\mathcal{A}^\star \;:\; \sigma(j)=pis} [\mathbf{M}_\sigma^{-1}\mathbf{P}(s), j]^\star$$

where $\mathbf{P} : \{1,2,3\}^\star \to \mathbb{Z}_+^3$, $w \mapsto (|w|_1, |w|_2, |w|_3)$.

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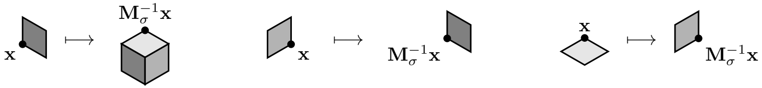
Example: $\mathbf{E}_1^*(\sigma)$ for $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$

$$\begin{aligned}
[\mathbf{x}, 1]^\star &\mapsto \mathbf{M}_\sigma^{-1}\mathbf{x} + [(1,0,-1),1]^\star \cup [(0,1,-1),2]^\star \cup [(0,0,0),3]^\star \\
[\mathbf{x}, 2]^\star &\mapsto \mathbf{M}_\sigma^{-1}\mathbf{x} + [(0,0,0),1]^\star \\
[\mathbf{x}, 3]^\star &\mapsto \mathbf{M}_\sigma^{-1}\mathbf{x} + [(0,0,0),2]^\star
\end{aligned}$$



Combinatorial tools: $\mathrm{E}_1^*(\sigma)$ dual substitutions

$\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

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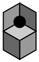
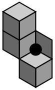
$\mathbf{E}_1^*(\sigma)($$)$



Combinatorial tools: $\mathbf{E}_1^*(\sigma)$ dual substitutions

$\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

$\mathbf{E}_1^*(\sigma)^2($$)$



Combinatorial tools: $\mathrm{E}_1^*(\sigma)$ dual substitutions

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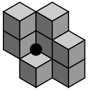
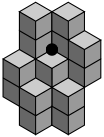
$\mathbf{E}_1^*(\sigma)^3($$)$



Combinatorial tools: $\mathbf{E}_1^*(\sigma)$ dual substitutions

$\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

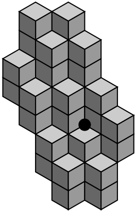
$\mathbf{E}_1^*(\sigma)^4($$)$



Combinatorial tools: $\mathrm{E}_1^*(\sigma)$ dual substitutions

$\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

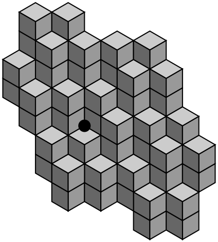
$\mathbf{E}_1^*(\sigma)^5($$)$



Combinatorial tools: $\mathrm{E}_1^*(\sigma)$ dual substitutions

$\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

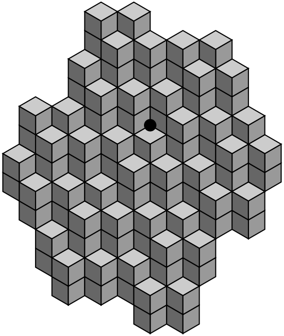
$\mathbf{E}_1^*(\sigma)^6($$)$



Combinatorial tools: $\mathrm{E}_1^*(\sigma)$ dual substitutions

$\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

$\mathbf{E}_1^*(\sigma)^7($$)$



Combinatorial tools: $\mathrm{E}_1^*(\sigma)$ dual substitutions

$\sigma : 1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$

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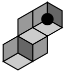
$\mathbf{E}_1^*(\sigma)($ ⬡ $)$



Combinatorial tools: $\mathrm{E}_1^*(\sigma)$ dual substitutions

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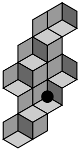

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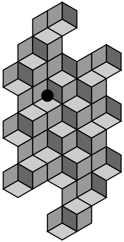

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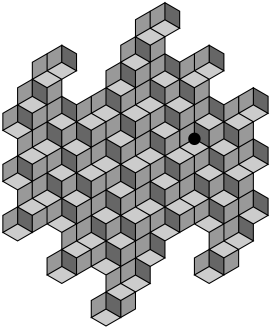

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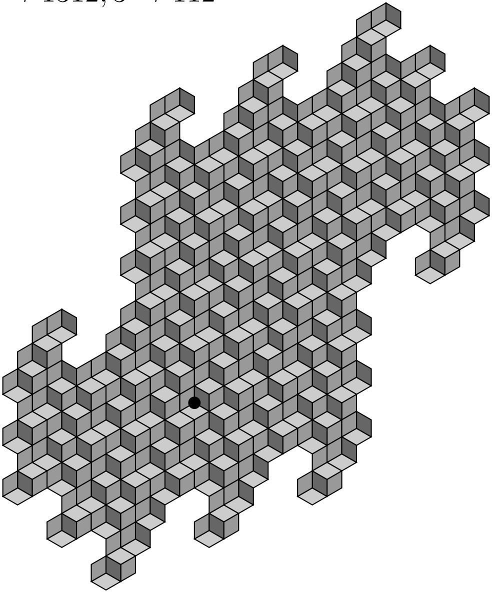

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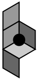

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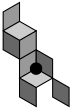

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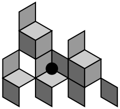

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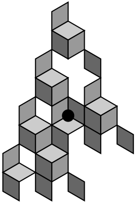

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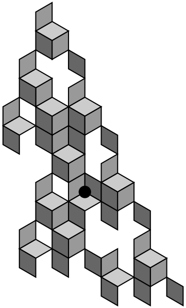

Combinatorial tools: $\mathrm{E}_1^*(\sigma)$ dual substitutions

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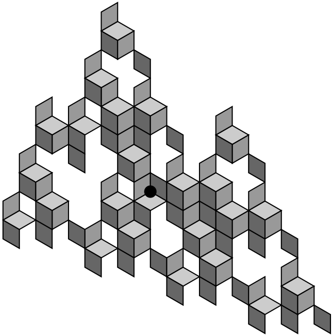
$\mathbf{E}_1^*(\sigma)^6($$)$



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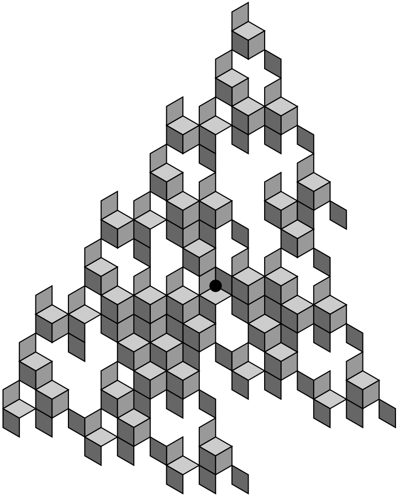
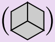
$\sigma : 1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$

$\mathbf{E}_1^*(\sigma)^7($  $)$



Combinatorial tools: $\mathrm{E}_1^*(\sigma)$ dual substitutions

Question: How do the patterns $\mathbf{E}_1^*(\sigma)^n($$)$ grow?

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Question: How do the patterns $\mathbf{E}_1^*(\sigma)^n($$)$ grow?

Theorem [Arnoux-Rauzy, Fernique]

- They grow within a discrete plane
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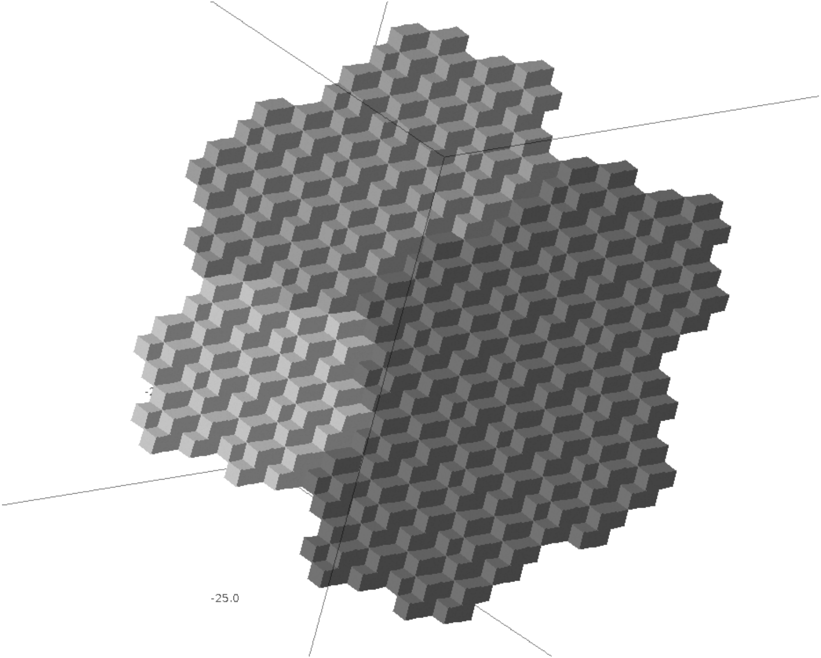
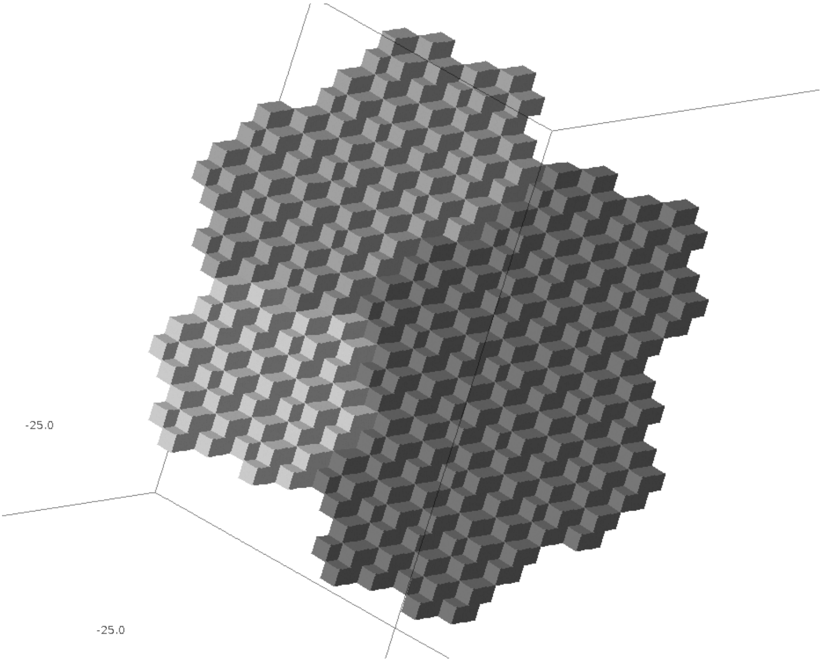

-25.0


-25.0
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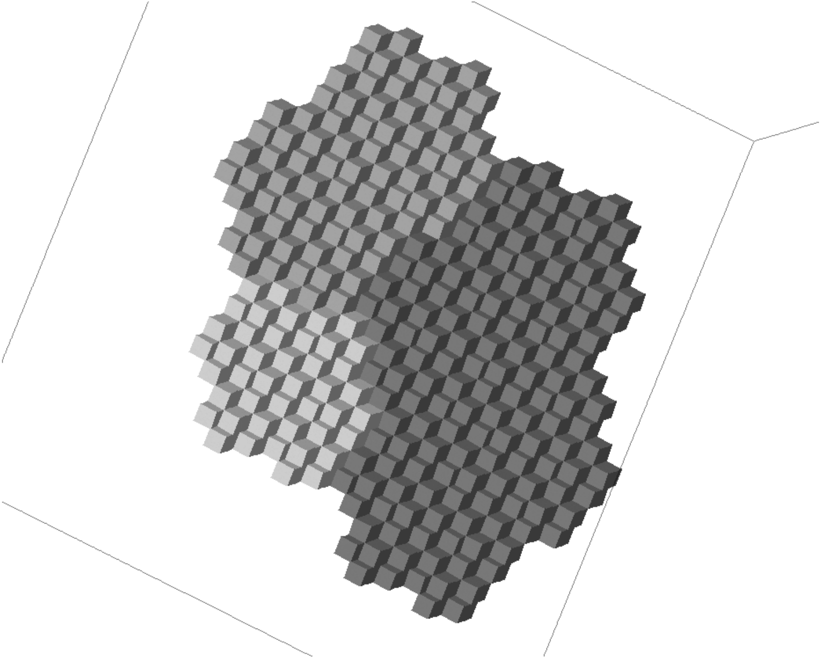

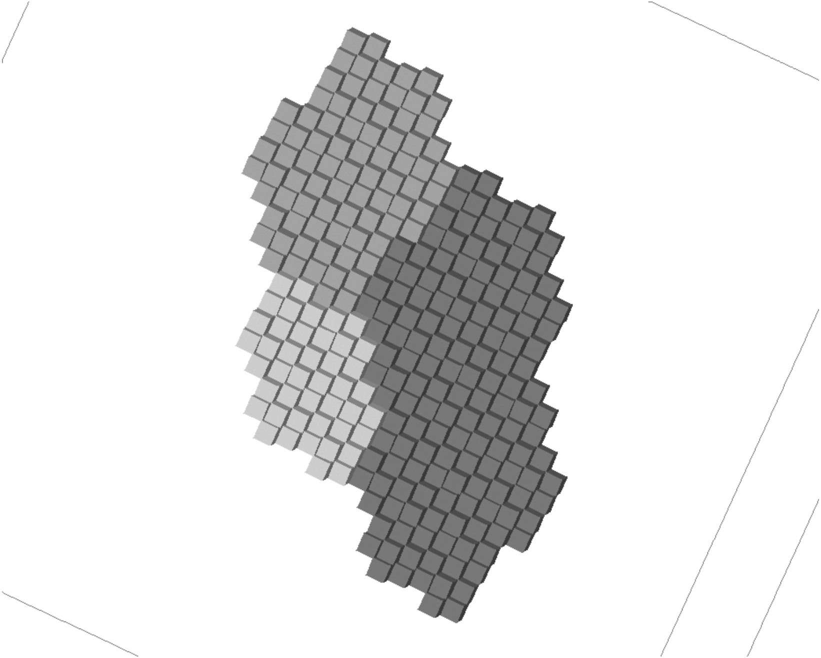

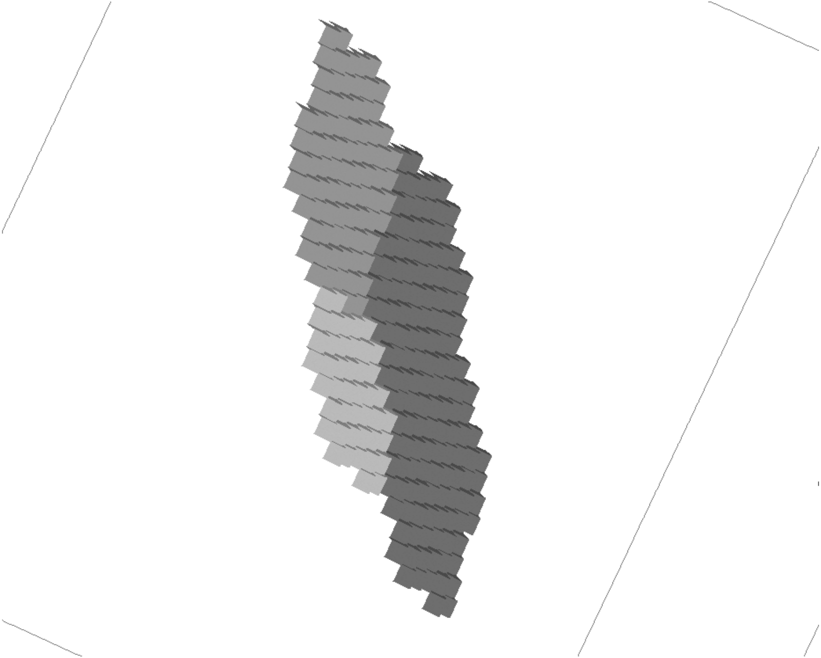

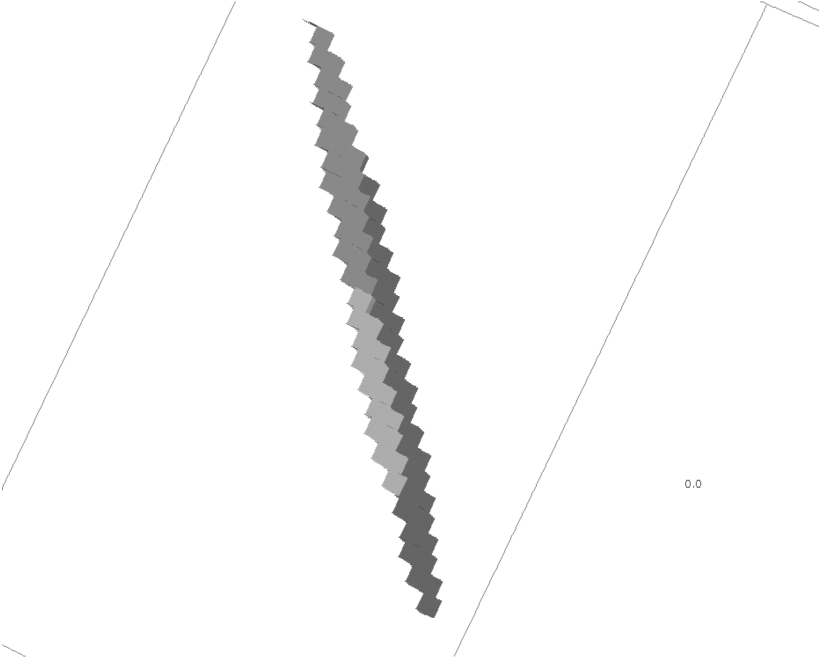
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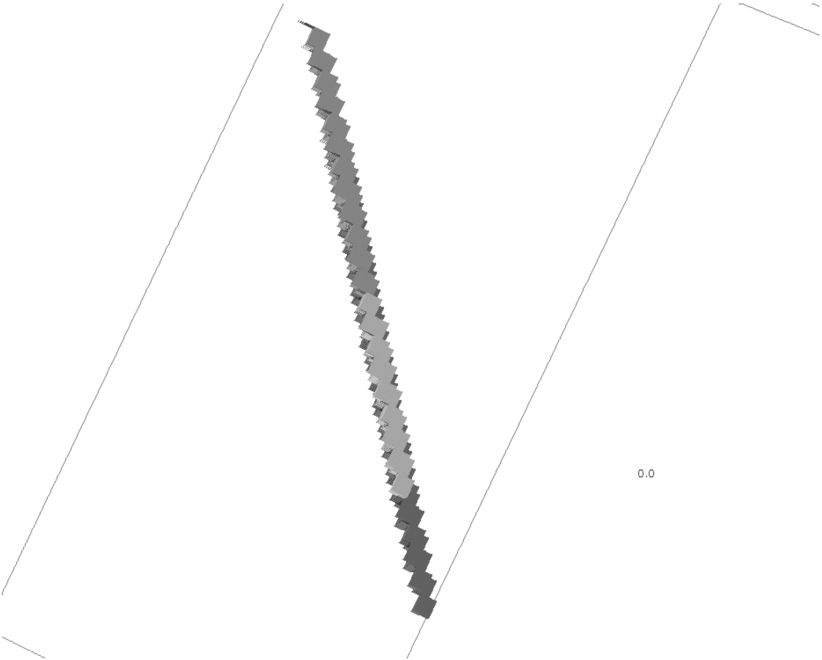

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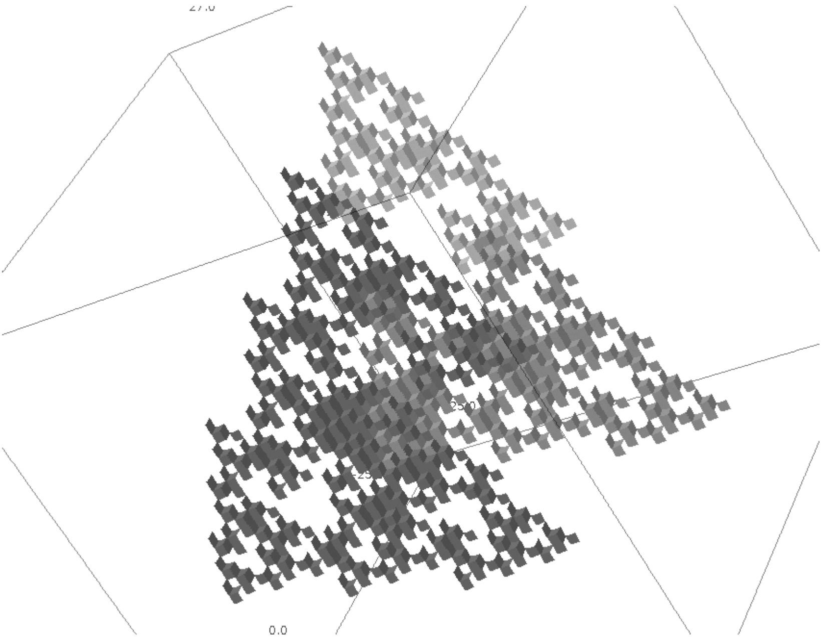
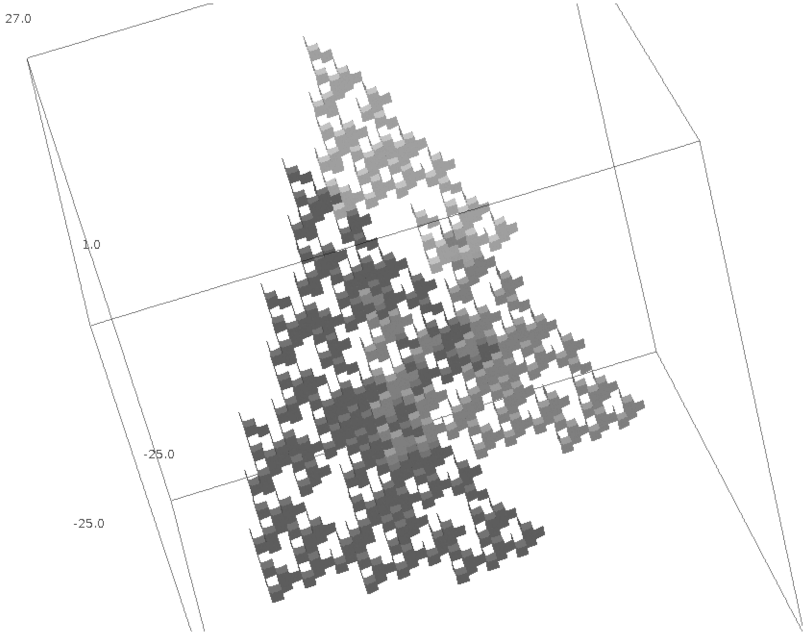
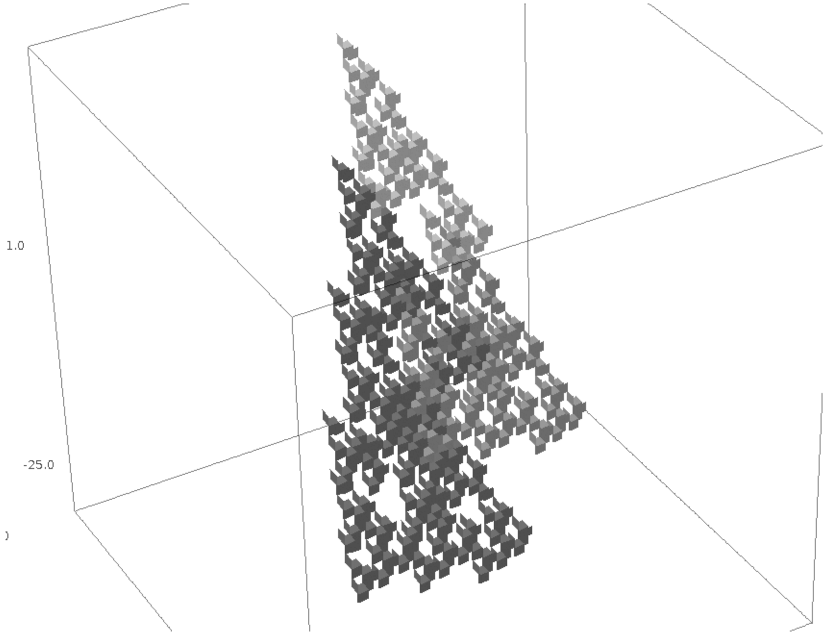
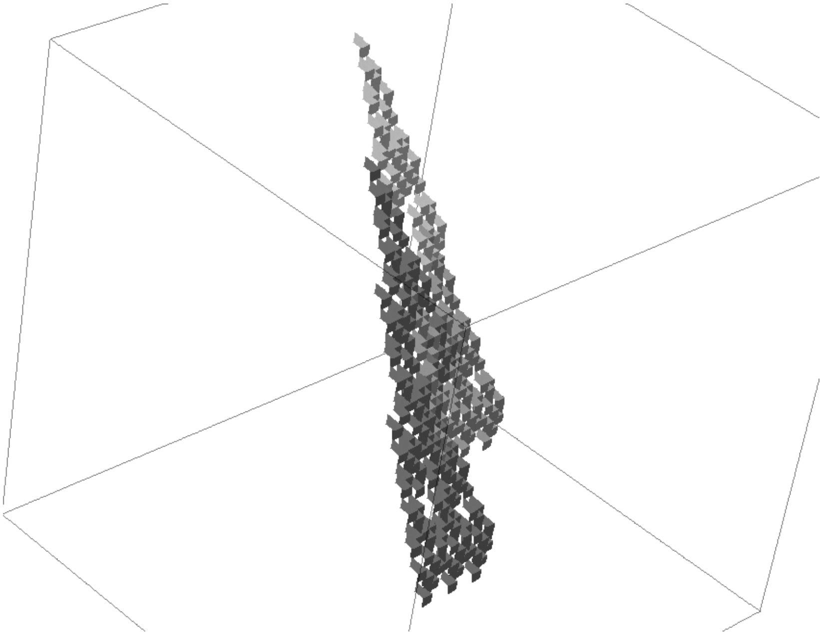


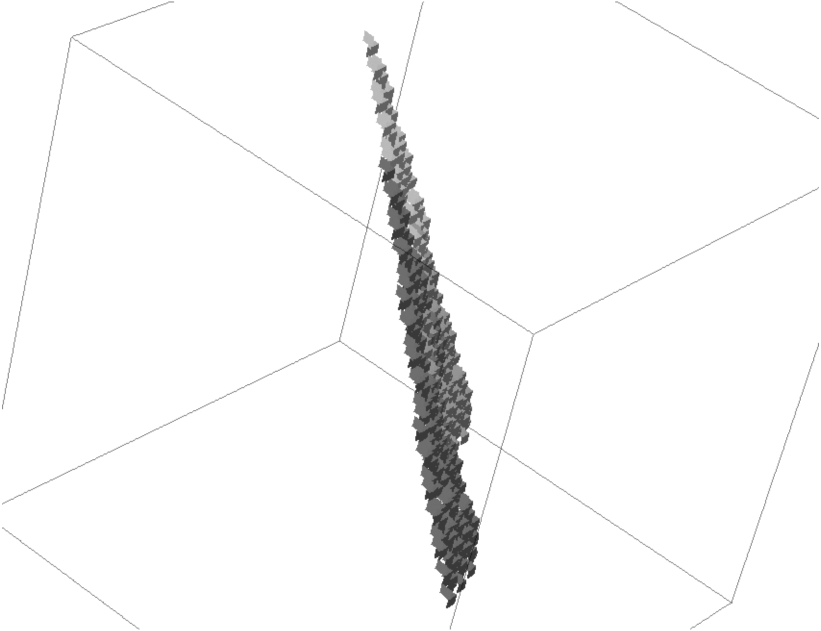
27.0
0.0

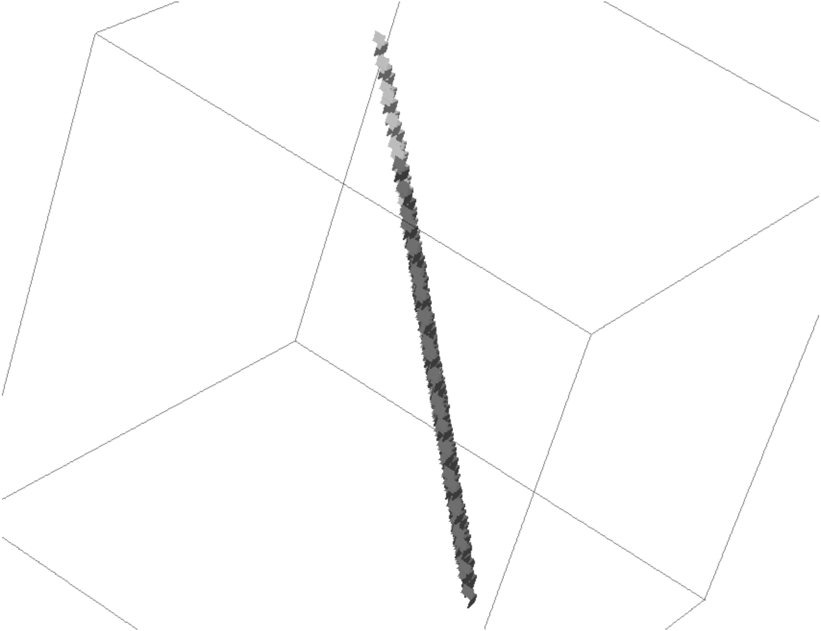

27.0
1.0
-25.0
-25.0



1.0
-25.0







Combinatorial tools: $\mathrm{E}_1^*(\sigma)$ dual substitutions

Question: How do the patterns $\mathbf{E}_1^*(\sigma)^n($$)$ grow?

Theorem [Arnoux-Rauzy, Fernique]

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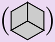
Question: How do the patterns $\mathbf{E}_1^*(\sigma)^n($$)$ grow?

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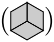
➥ **Advantage:** arbitrary products $\mathbf{E}_1^*(\sigma_{i_1})\cdots\mathbf{E}_1^*(\sigma_{i_n})($$)$

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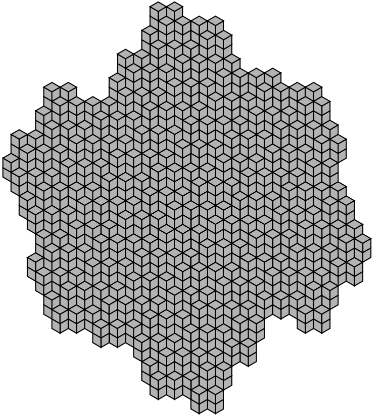
➥ **Advantage:** arbitrary products $\mathbf{E}_1^*(\sigma_{i_1}) \cdots \mathbf{E}_1^*(\sigma_{i_n})($$)$

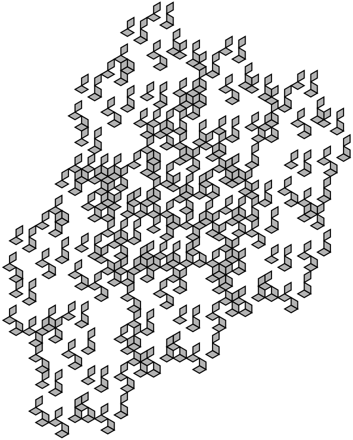
Theorem [Ito-Rao 2006]

σ satisfies the super-coincidence condition
$\Longleftrightarrow \mathbf{E}_1^*(\sigma)^n($$)$ covers arbitrarily large discs

Covering arbitrarily large discs

$1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ vs. $1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 12$



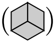


Not always obvious

Covering arbitrarily large discs

- Back to Brun substitutions
- Let $(i_n) \in \{1, 2, 3\}^{\mathbb{N}}$ such that $i_n = 3$ infinitely many often

Covering arbitrarily large discs

- Back to Brun substitutions
- Let $(i_n) \in \{1,2,3\}^{\mathbb{N}}$ such that $i_n = 3$ infinitely many often
- **Goal:** the patterns $\mathbf{E}_1^*(\sigma_{i_1})\mathbf{E}_1^*(\sigma_{i_2})\cdots\mathbf{E}_1^*(\sigma_{i_n})($$)$ cover arbitrarily large discs as $n \to \infty$.

Covering arbitrarily large discs

$(i_n) = 333\ldots$, iterating $\mathbf{E}_1^*(\sigma_{i_1}) \cdots \mathbf{E}_1^*(\sigma_{i_n})($$)$

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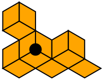

Covering arbitrarily large discs

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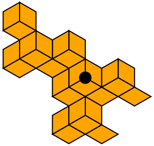
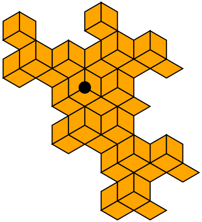


Covering arbitrarily large discs

$(i_n) = 333\ldots$, iterating $\mathbf{E}_1^*(\sigma_{i_1}) \cdots \mathbf{E}_1^*(\sigma_{i_n})($$)$



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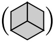
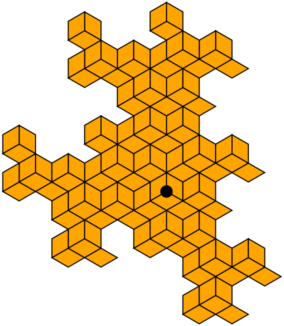
$(i_n) = 333\ldots$, iterating $\mathbf{E}_1^*(\sigma_{i_1})\cdots\mathbf{E}_1^*(\sigma_{i_n})($$)$



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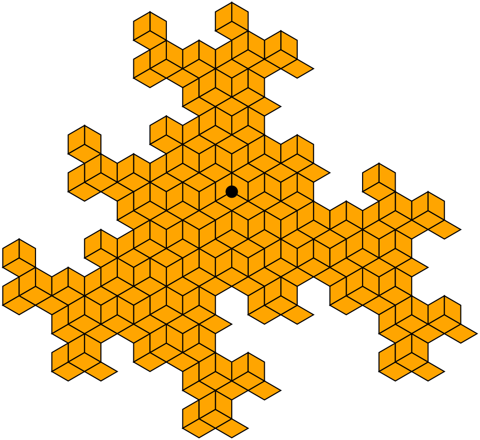
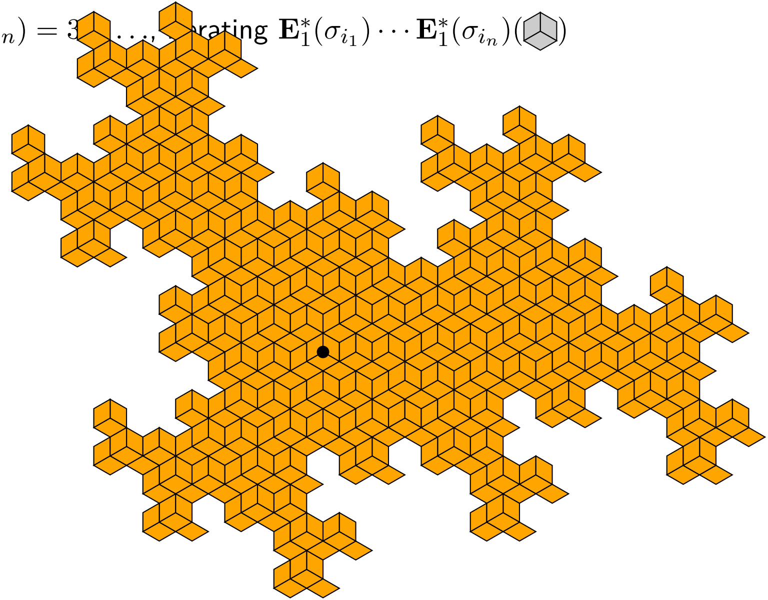


Covering arbitrarily large discs

$(i_n) = 3$ …, …erating $\mathbf{E}_1^*(\sigma_{i_1}) \cdots \mathbf{E}_1^*(\sigma_{i_n})$()



Covering arbitrarily large discs

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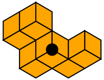

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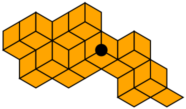

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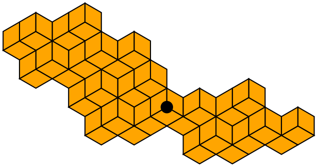

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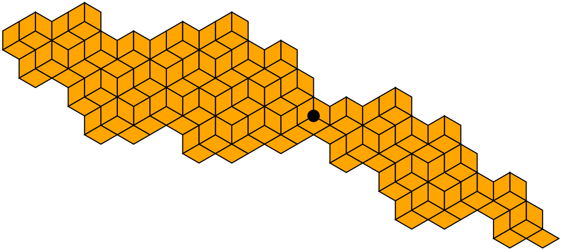

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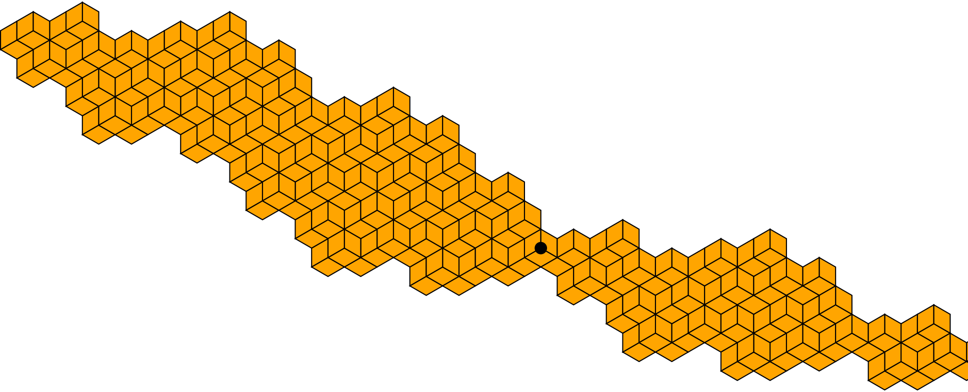
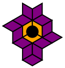


Covering arbitrarily large discs: **the annulus property**



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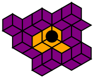

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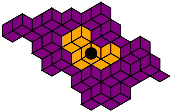

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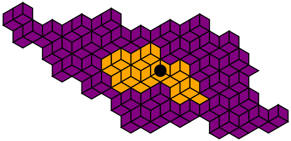

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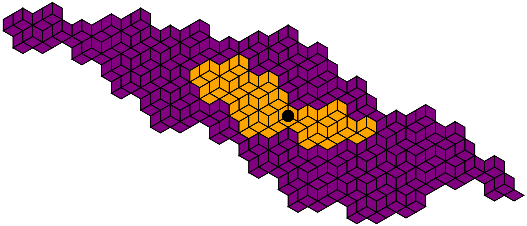

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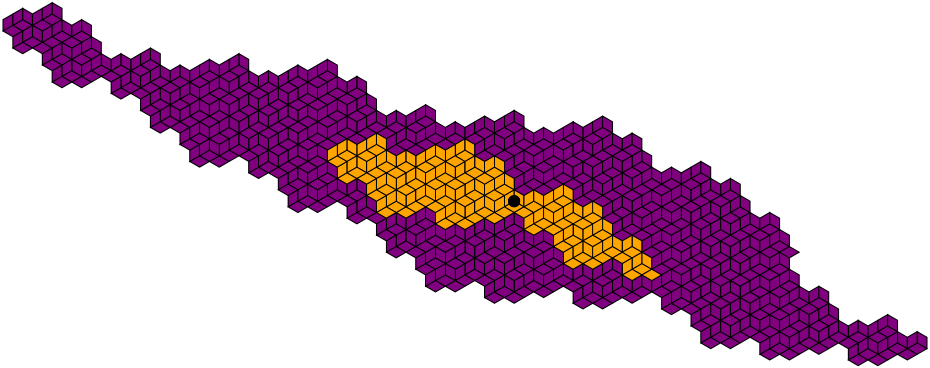

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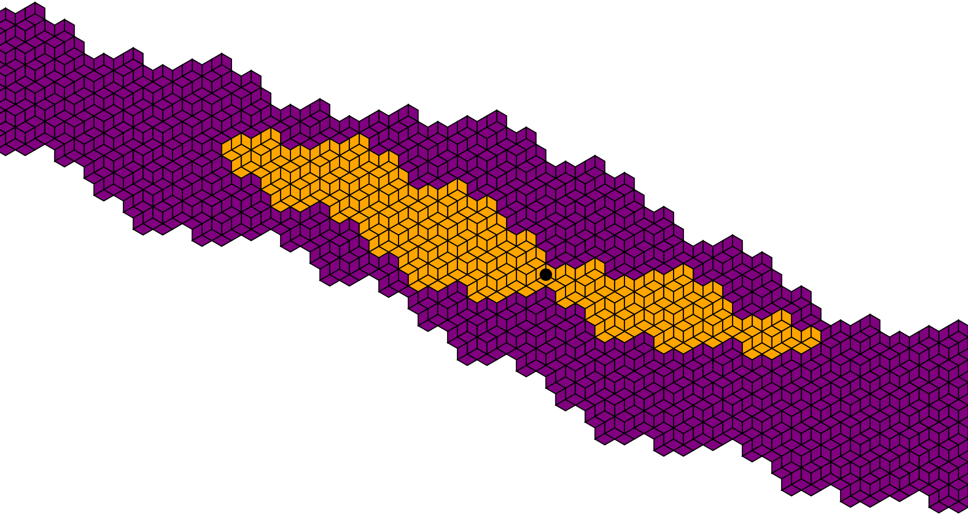
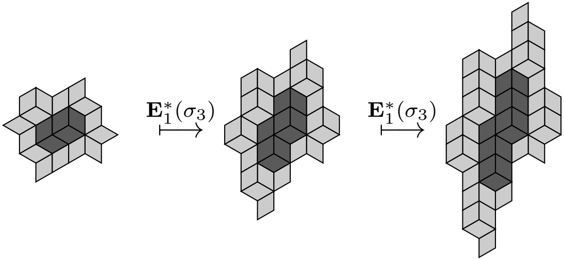


Covering arbitrarily large discs: covering properties

Unfortunately the annulus property **doesn't always hold**:



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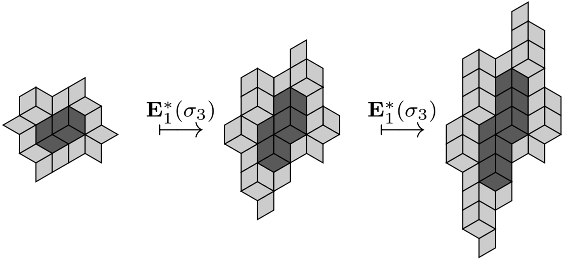

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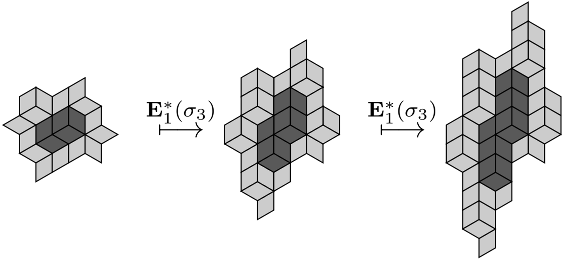

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- Stronger assumptions on A: **covering properties**.

Covering arbitrarily large discs: **covering properties**

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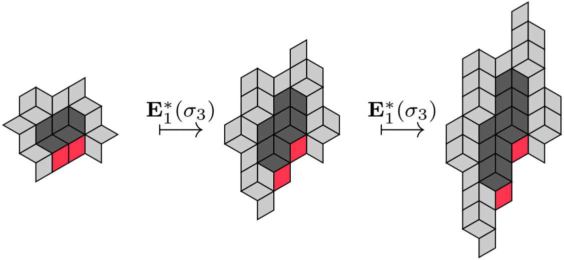

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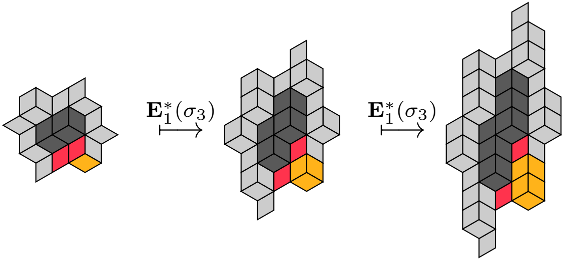

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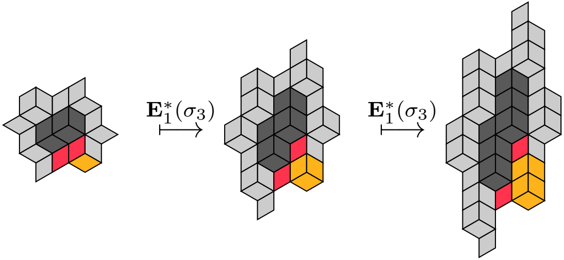
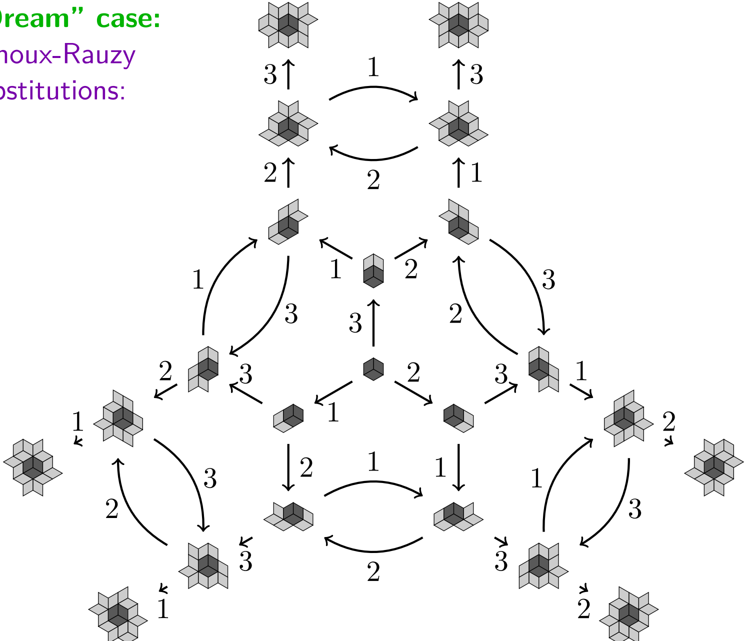


- We have to be more careful.
- Stronger assumptions on A: **strong covering properties**.
- Annulus property originally in [Ito-Ohtsuki 94]
  for Jacobi-Perron substitutions

Generating a first seed: generation graphs

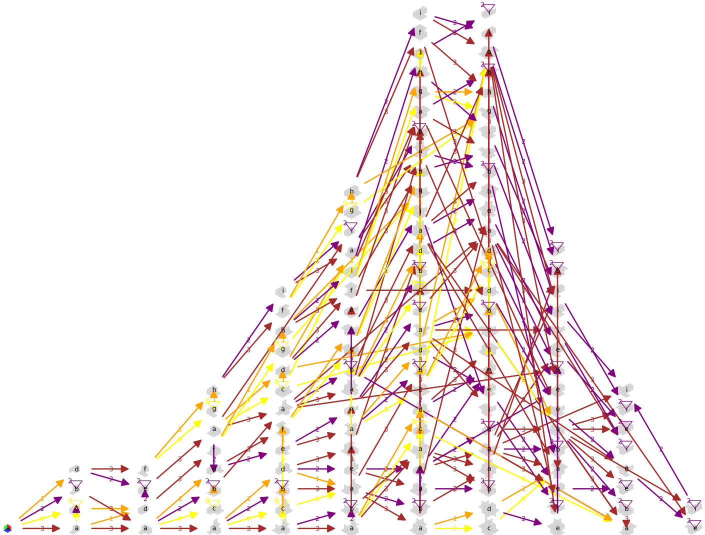
"Dream" case:
Arnoux-Rauzy substitutions:



Generating a first seed: generation graphs

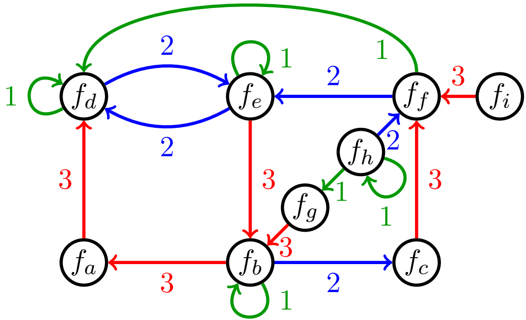
Bad approach:

Brun, Jacobi-Perron substitutions:



Generating a first seed: generation graphs

New approach:



- Full understanding of the bad language
- Allows to easily compute the finite seed
- Graph obtained algorithmically

Main result

Let $(i_n)_{n\in\mathbb{N}} \in \{1,2,3\}^{\mathbb{N}}$ with infinitely many 3's.

Theorem [Berthé-Bourdon-J-Siegel]

The patterns $\mathbf{E}_1^*(\sigma_{i_1})\cdots\mathbf{E}_1^*(\sigma_{i_n})($$)$

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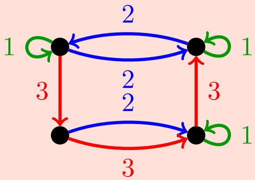
   $\iff \exists$ an infinite path $\cdots \xrightarrow{i_2} \bullet \xrightarrow{i_1} \bullet$ in the graph:



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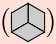
The patterns $\mathbf{E}_1^*(\sigma_{i_1})\cdots\mathbf{E}_1^*(\sigma_{i_n})($$)$

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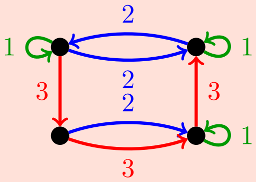

- **New combinatorial methods:** (strong) covering properties
- **New algorithmic tools:** generation graphs

Plan

- Substitutions: introduction

I. Pisot substitutions

- Dynamics of Pisot substitutions
- Combinatorial tools: dual substitutions
- **Applications**

Applications: dynamics

Theorem [Berthé-Bourdon-J-Siegel]

For every valid Brun product $\sigma = \sigma_{i_1} \cdots \sigma_{i_n}$, we have

$$(X_\sigma\,, \text{shift}) \;\cong\; (\mathbb{T}^2\,, \text{translation})$$

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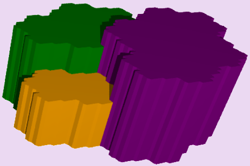
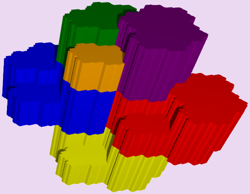
Corollary [Praggastis, Ito-Rao, Siegel]

We can construct explicit Markov partitions of $(\mathbb{T}^3, x \mapsto \mathbf{M}x)$ for every $\mathbf{M} = \mathbf{M}_\sigma$, where $\sigma = \sigma_{i_1} \cdots \sigma_{i_n}$.





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For every cubic real extension $\mathbb{K}$ of $\mathbb{Q}$, there exists $\alpha, \beta \in \mathbb{K}$ such that $(\mathbb{T}^2, x \mapsto x + (\alpha, \beta))$ is semi-conjugate with the dynamics of some product $\sigma = \sigma_{i_1} \cdots \sigma_{i_n}$.

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- **Towards S-adic sequences** [Berthé-Steiner-Thuswaldner]

Applications: discrete geometry, combinatorics on words

- **Generalizing Sturmian sequences:**
  For **any** $\mathbf{v} = (1, \alpha, \beta)$, we can generate the discrete plane $\mathbf{\Gamma_v}$ using the Brun algorithm and $\mathbf{E}_1^*$ substitutions.

Applications: discrete geometry, combinatorics on words

- **Generalizing Sturmian sequences:**
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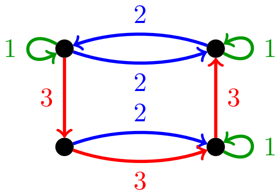
- **Critical connectedness of arithmetical discrete planes**
  [Berthé-Jamet-J-Provençal]

Applications: topology of Rauzy fractals

- **0 is an inner point** of the Rauzy fractal of $\sigma_{i_1} \cdots \sigma_{i_n}$
  $\Longleftrightarrow i_1 \cdots i_n$ labels a loop in the graph:



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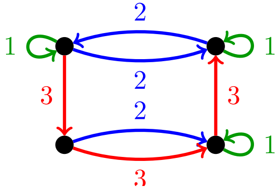

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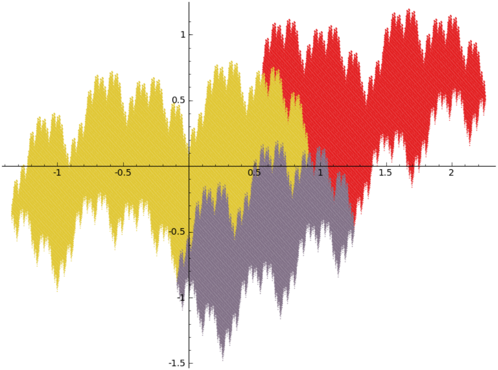

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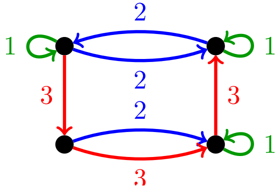

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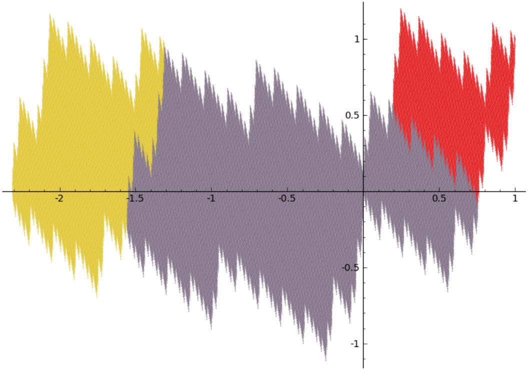

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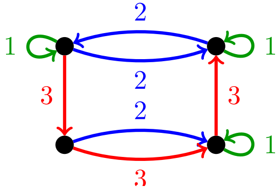

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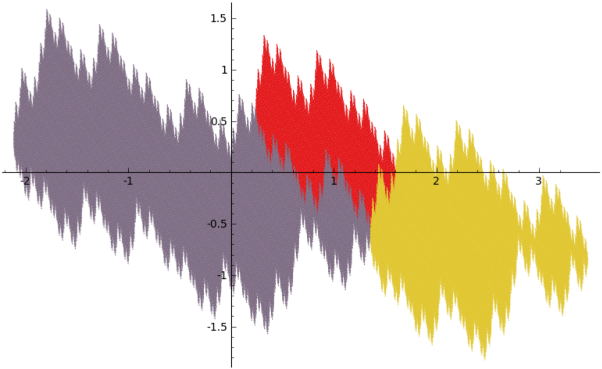

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- **0 is an inner point** of the Rauzy fractal of $\sigma_{i_1} \cdots \sigma_{i_n}$ $\iff i_1 \cdots i_n$ labels a loop in the graph.
- **Connectedness** of the Rauzy fractals of the $\sigma_{i_1} \cdots \sigma_{i_n}$.
- **Question:** For which products $\sigma_{i_1} \cdots \sigma_{i_n}$ is the fractal simply connected?

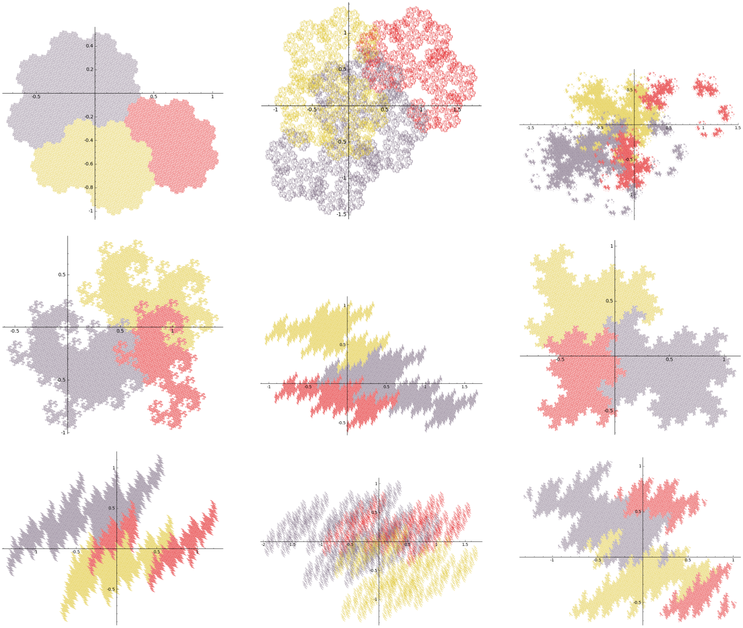
Plan

- Substitutions: introduction

I. Pisot substitutions

- Dynamics of Pisot substitutions
- Combinatorial tools: dual substitutions
- Applications
- **Topology: Rauzy fractals with countable fundamental group**

Countable fundamental groups of Rauzy fractals



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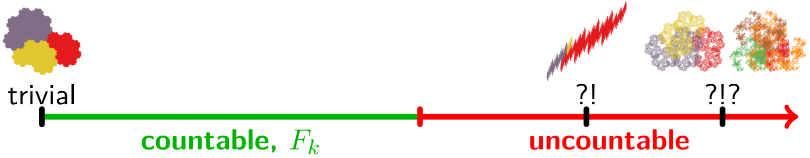

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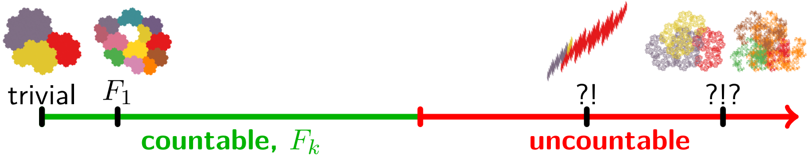

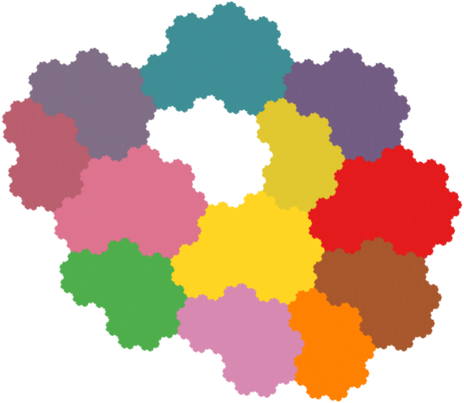

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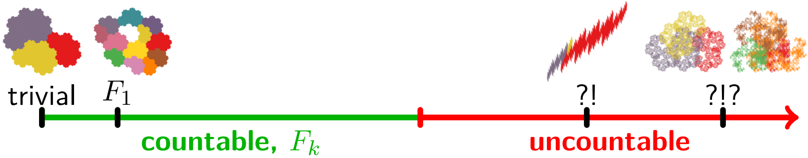
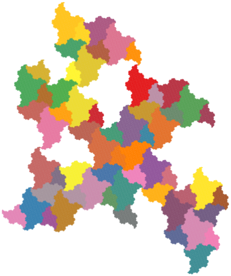
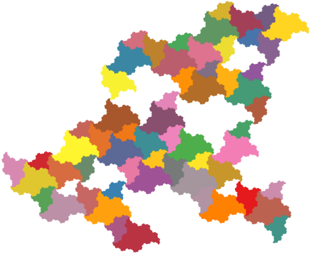


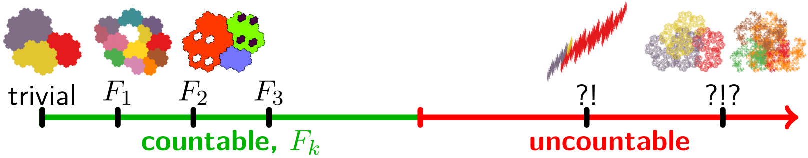


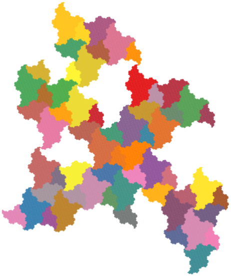
F_2



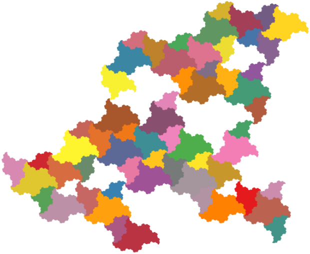
F_6

Countable fundamental groups of Rauzy fractals





F_2



F_6

Countable fundamental groups of Rauzy fractals

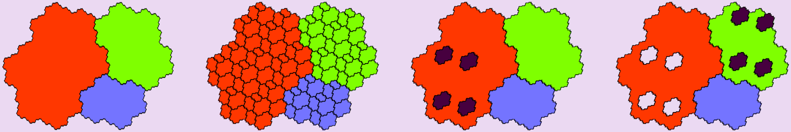
Theorem [J-Loridant-Luo]

For **every** $k \in \mathbb{N}$, there exists σ such that

- $\pi_1(\mathcal{T}_\sigma(1)) \cong F_k$
- $\pi_1(\mathcal{T}_\sigma) \cong F_k$

Moreover, σ can be taken with 4 letters only.

$k = 4$



σ $\quad$ σ^6 $\quad$ splitting τ $\quad$ $\rho_{42}^{-1}\tau\rho_{42}$

Countable fundamental groups of Rauzy fractals

Theorem [J-Loridant-Luo]

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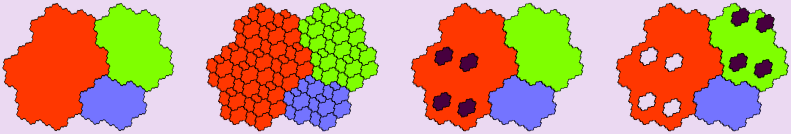

σ σ^6 splitting τ $\rho_{42}^{-1}\tau\rho_{42}$

➥ Symbolic operations on σ: state splittings, free group aut.

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- Combinatorial tools: dual substitutions
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II. Stepping back from the Pisot case, (un)decidability

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- **Combinatorial substitutions**

Combinatorial substitutions

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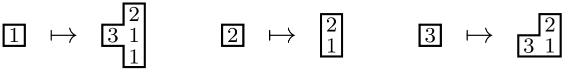

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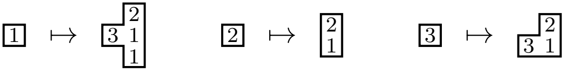

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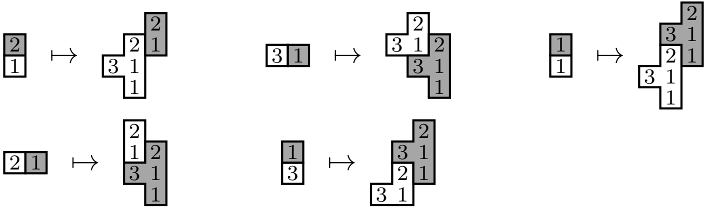

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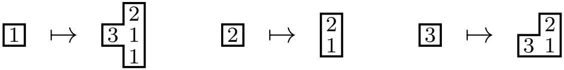

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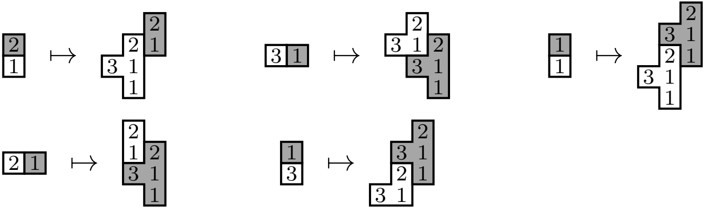

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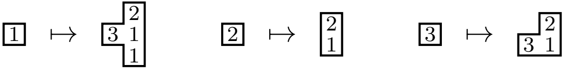

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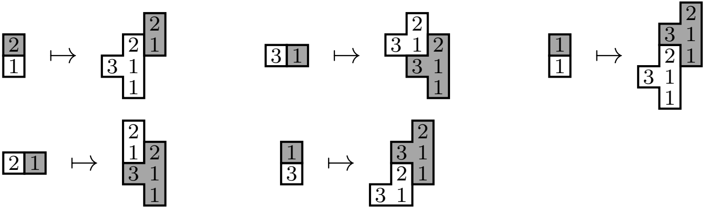

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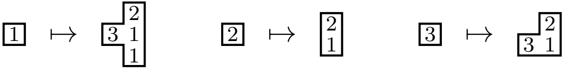

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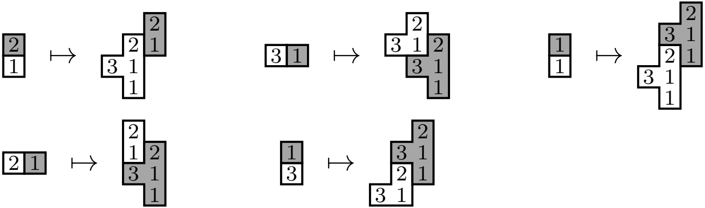

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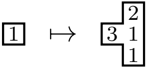

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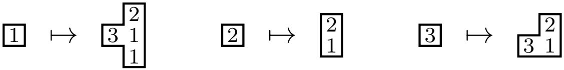

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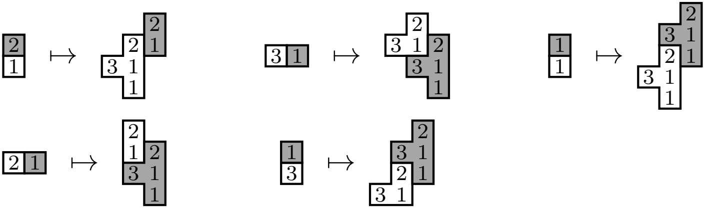

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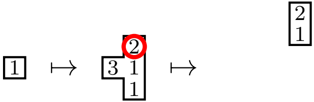

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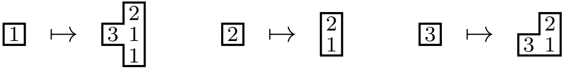
Base rule: 

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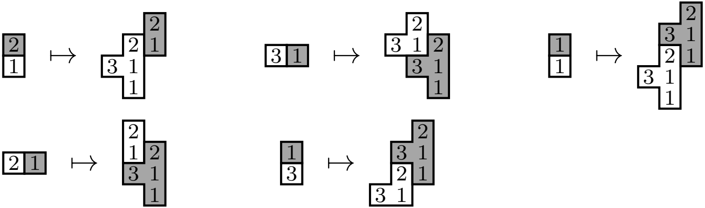

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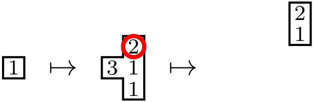

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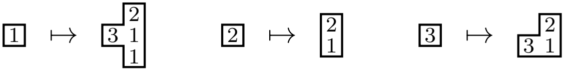

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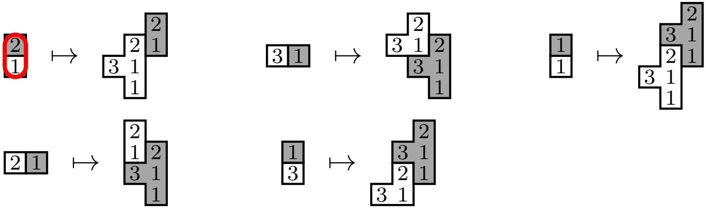

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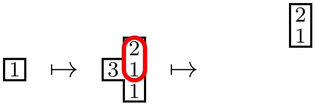

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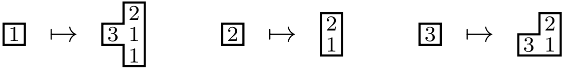

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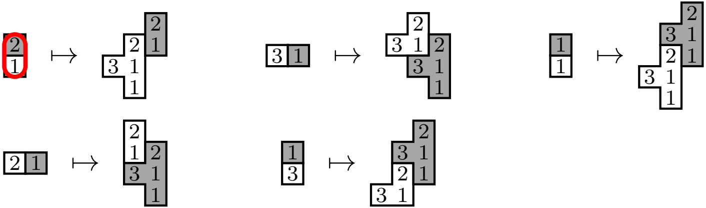

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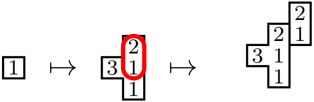

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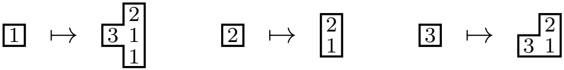
Base rule: 

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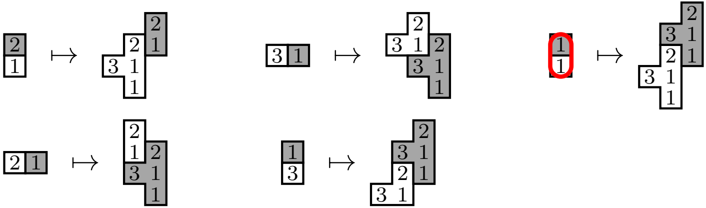

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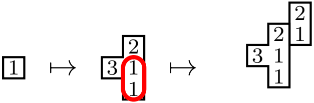

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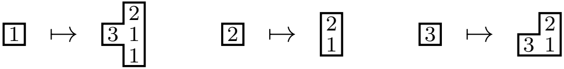

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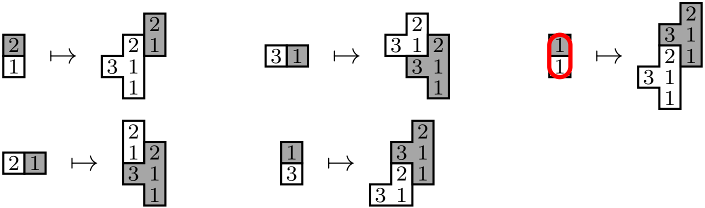

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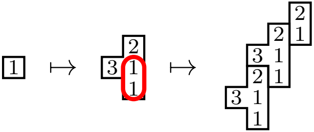

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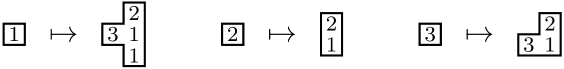

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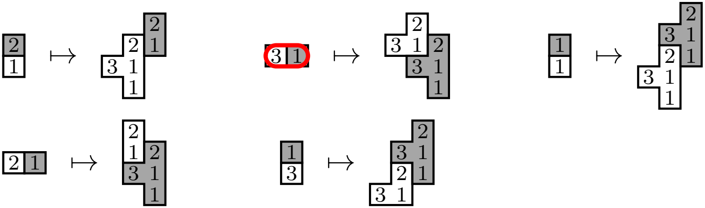

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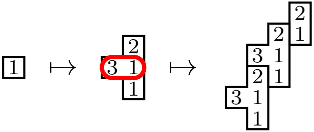

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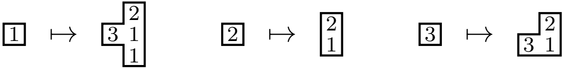

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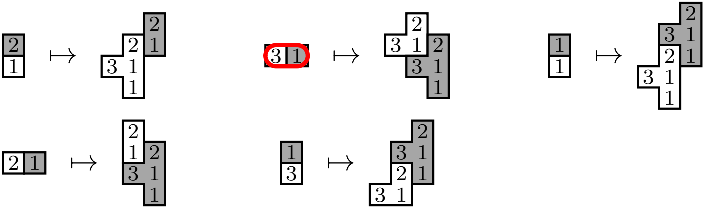

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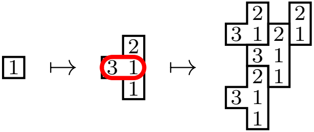

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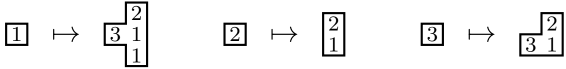

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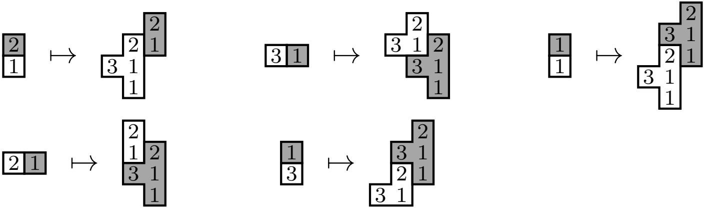

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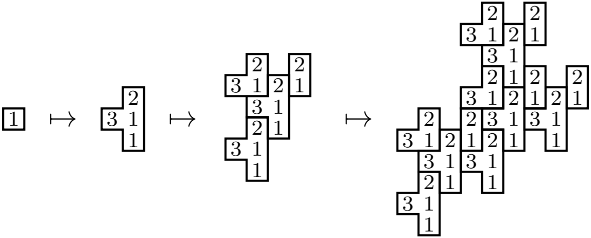

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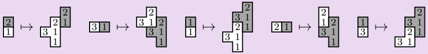
2. concatenation rules ().

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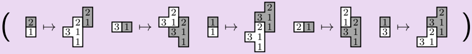
2. concatenation rules .

- Very general definition

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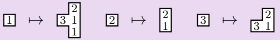
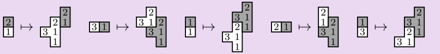
1. base rules (),
2. concatenation rules ().

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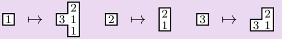
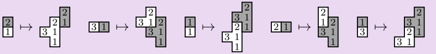
1. base rules (),
2. concatenation rules ().

- Very general definition
- **But:**
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  - Can produce **overlapping** patterns.
- Can we detect such problems?

Combinatorial substitutions

Theorem [J-Kari]

It is undecidable if a combinatorial substitution is consistent or non-overlapping

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Theorem [J-Kari]

This can be decided for domino-complete substitutions.

- Provides combinatorial, "automatic" proofs for some previously known results about $\mathbf{E}_1^*$ substitutions

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- **Affine iterated function systems**

Affine iterated function systems

Iterated function system:

- Let $f_1, \ldots, f_n : \mathbb{R}^d \to \mathbb{R}^d$ be contracting maps
- Uniquely defines a compact set $X \subseteq \mathbb{R}^d$ such that $X = f_1(X) \cup \cdots \cup f_n(X)$

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Example: Cantor set $X = f_1(X) \cup f_2(X) \subseteq \mathbb{R}$

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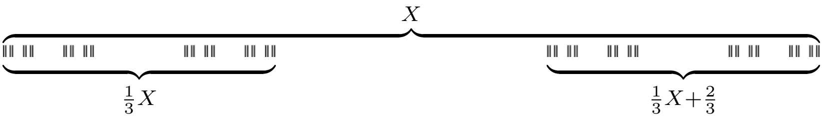

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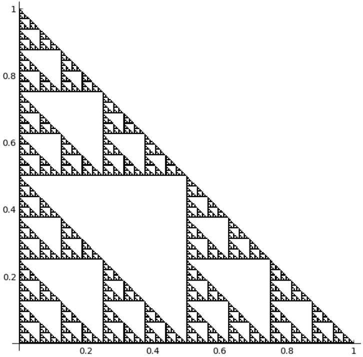
Example: Sierpiński triangle $X = f_1(X) \cup f_2(X) \cup f_3(X) \subseteq \mathbb{R}^2$

- $f_1 : x \mapsto \frac{1}{2}x$
- $f_2 : x \mapsto \frac{1}{2}x + \binom{1/2}{0}$
- $f_3 : x \mapsto \frac{1}{2}x + \binom{0}{1/2}$



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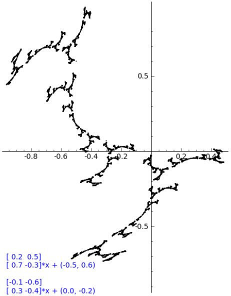

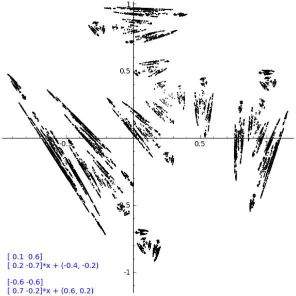

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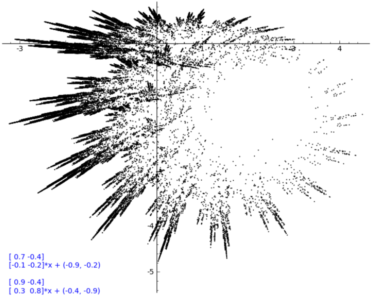

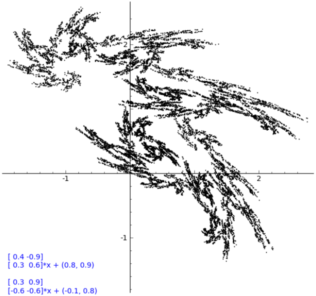

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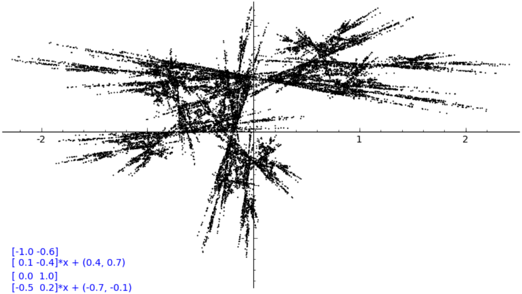

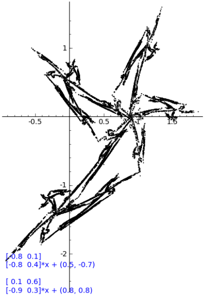

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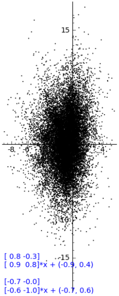

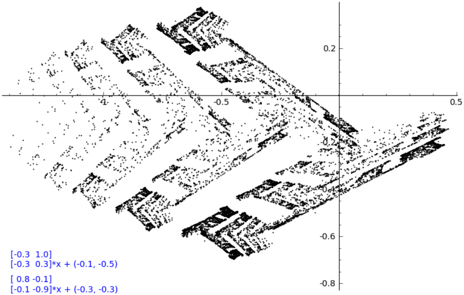

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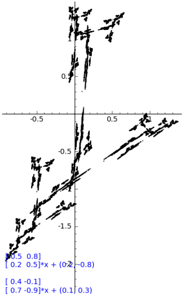
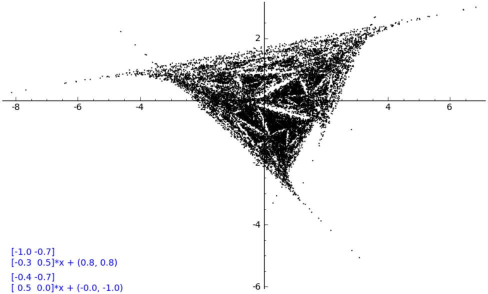
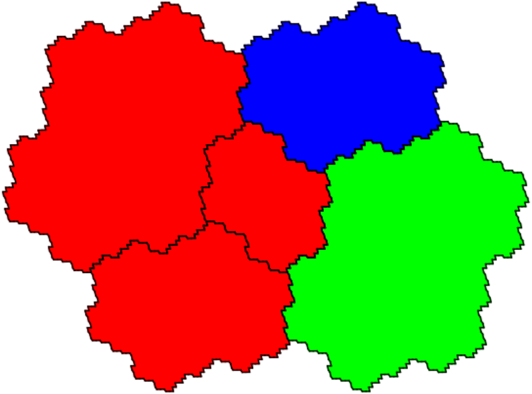




Affine iterated function systems

Example: Tribonacci fractal $\begin{cases} X_1 = f_1(X_1) \cup f_2(X_2) \cup f_3(X_3) \\ X_2 = f_4(X_1) \\ X_3 = f_5(X_2) \end{cases}$



Affine iterated function systems

- ☺ Results about **some** families (Bedford-McMullen carpets), "almost sure" formula for Hausdorff dimension (Falconer), . . .

Affine iterated function systems

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Question: given rational affine maps $f_1, \ldots, f_n$, is it **decidable** if:

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- X has Hausdorff dimension 2?
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➥ Mostly decidable for Rauzy fractals [Siegel-Thuswaldner]

Affine iterated function systems

Theorem [J-Kari 2013]

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Property of the d-tape automaton	Topological property of the attractor
Accepts a configuration with identical tape contents	Intersects the diagonal [Dube]
Is universal	Is equal to $[0,1]^d$
Has universal prefixes	Has nonempty interior
?	Is connected
?	Is totally disconnected
Compute language entropy	Compute Hausdorff dimension

Conclusion

I. Pisot substitutions

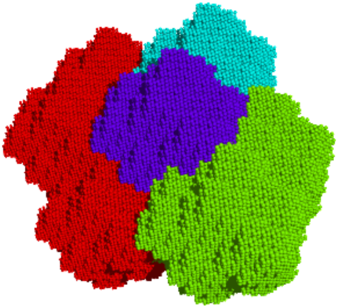
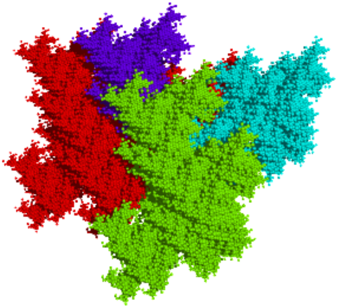
- Dynamics of Pisot substitutions
- Combinatorial tools: dual substitutions
- Applications
- Topology: Rauzy fractals with countable fundamental group

II. Stepping back from the Pisot case, (un)decidability

- Combinatorial substitutions
- Affine iterated function systems

Some perspectives

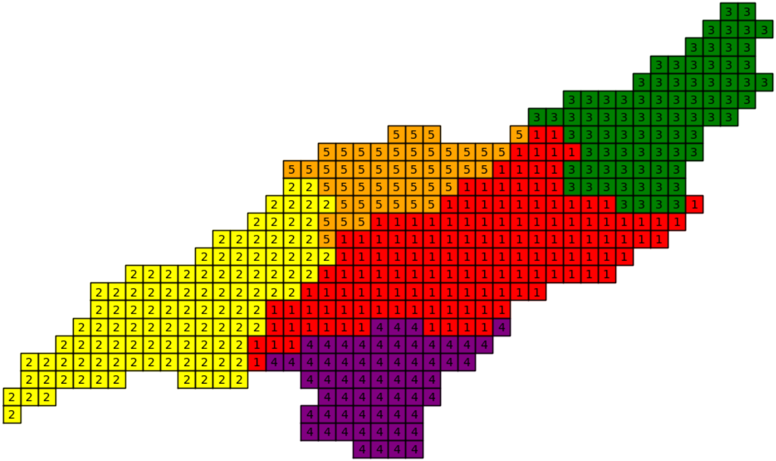
3D Rauzy fractals (Pisot β of degree ~~3~~ 4)

Difficulty: the boundary is a ~~curve~~

Some perspectives

$\mathbf{E}_1^*$-like combinatorial tools for **reducible substitutions**



Some perspectives

Symbolic dynamics

- Conjugacy problem for substitutions: decision problems
- Bratteli diagrams for 2D Sturmian sequences
- Higher-dimensional substitutions: decision problems

The end

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Thank you for your attention