Products of Pisot substitutions

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Joint work with **Valérie Berthé**, **Jérémie Bourdon**, **Anne siegel**

Journées SDA2
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Pisot substitutions

$$\sigma : \begin{cases} 1 & \mapsto & 12 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 1 \end{cases} \qquad \mathbf{M}_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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1

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12

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1213

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σ is Pisot:

- Real eigenvalue $\beta > 1$ of $\mathbf{M}_\sigma$ $\qquad \beta \approx 1.839$
- Conjugates $|\beta'|, |\beta''| \in\,]0,1[$ $\qquad \beta', \beta'' \approx -0.419 \pm 0.606i$

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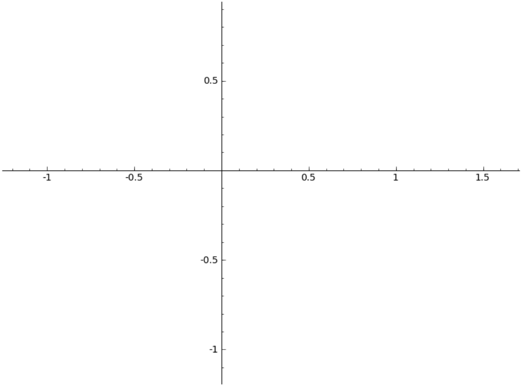
Action of M_σ on $\mathbb{R}^3$:

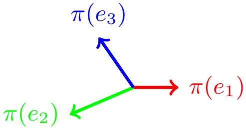
- **Expanding line** $\mathbb{E}$ spanned by eigenvector $\mathbf{u}_\beta$
- **Contracting plane** $\mathbb{P}$ spanned by eigenvectors $\mathbf{u}_{\beta'}, \mathbf{u}_{\beta''}$

Rauzy fractals

$1213121121312121312112 13\cdots$



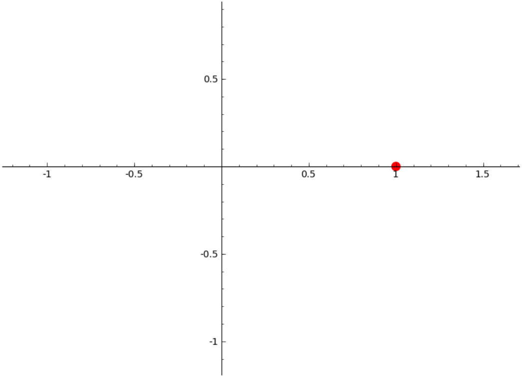
$\pi : \mathbb{R}^3 \to \mathbb{P}$,
projection along $\mathbb{E}$



Rauzy fractals

$121312112131212131211213\cdots$

$\pi(e_1)$



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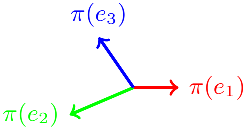

Rauzy fractals

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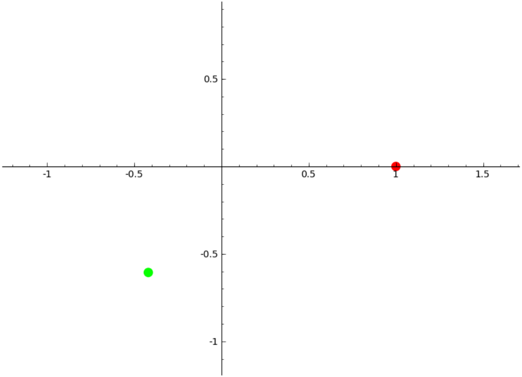
$\pi(e_1+e_2)$



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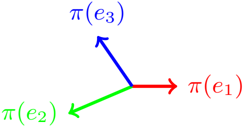

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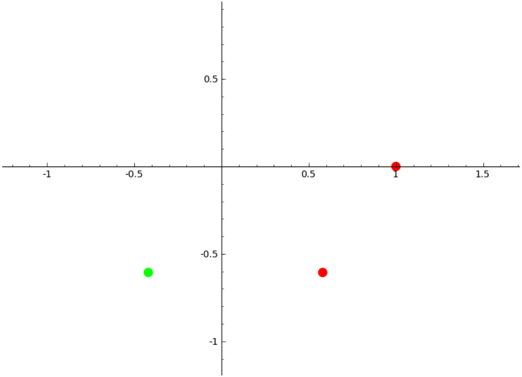
$\pi(e_1 + e_2 + e_1)$



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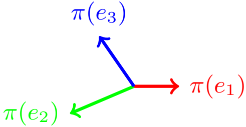

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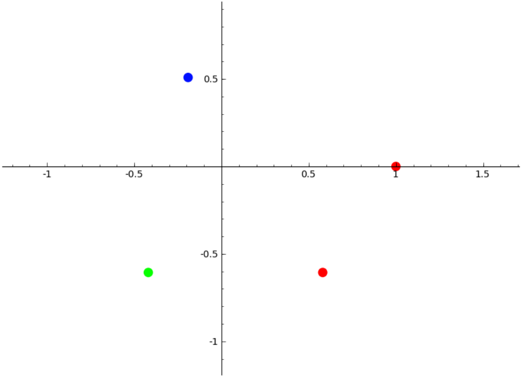

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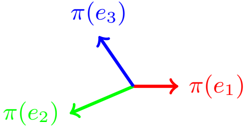

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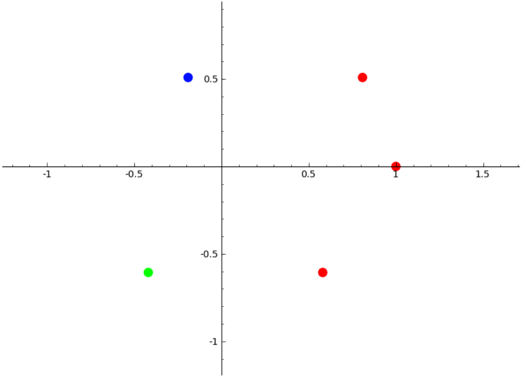

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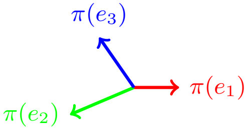

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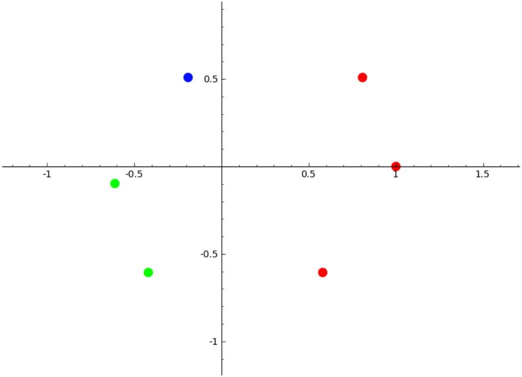

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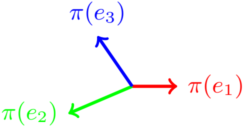

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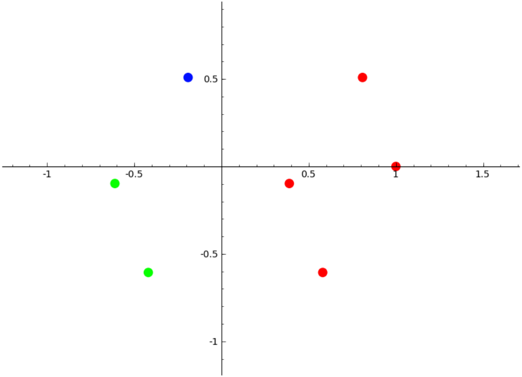

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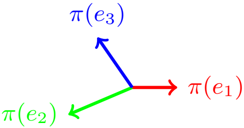

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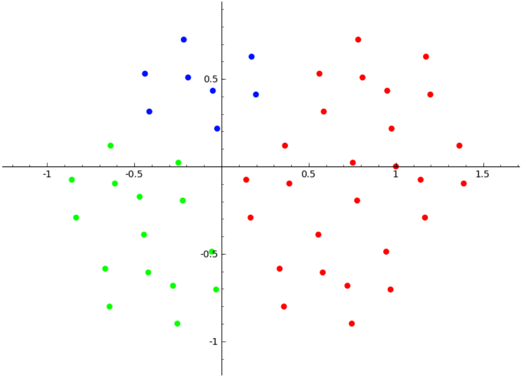

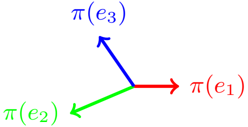

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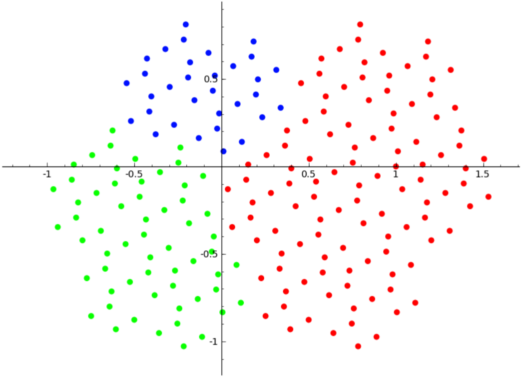

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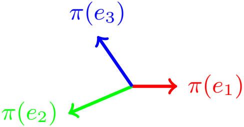

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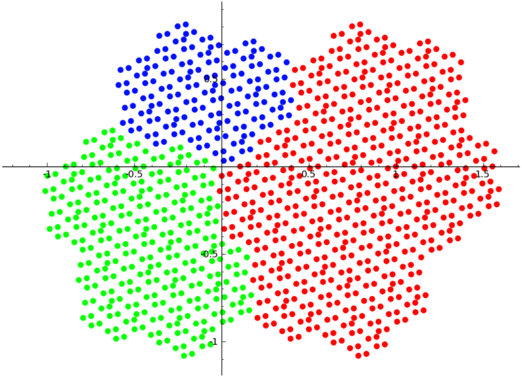

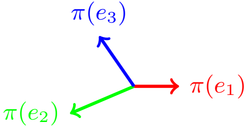

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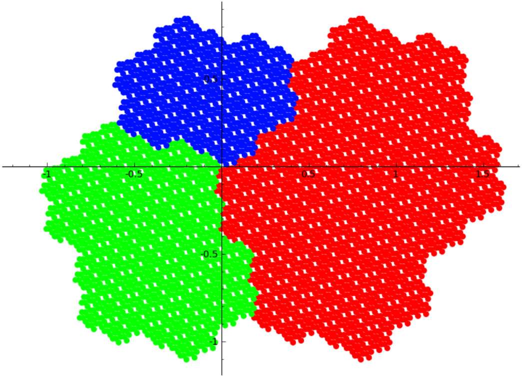

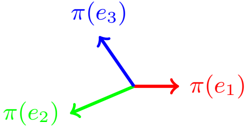

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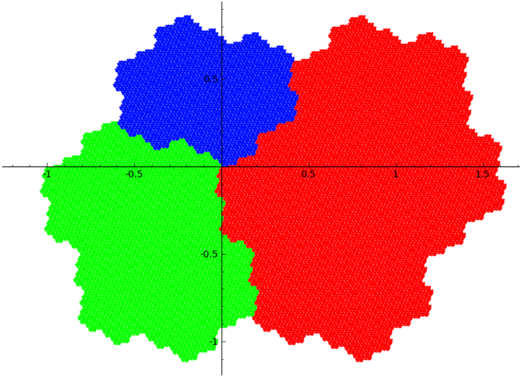

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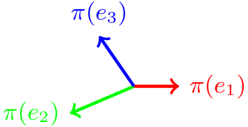
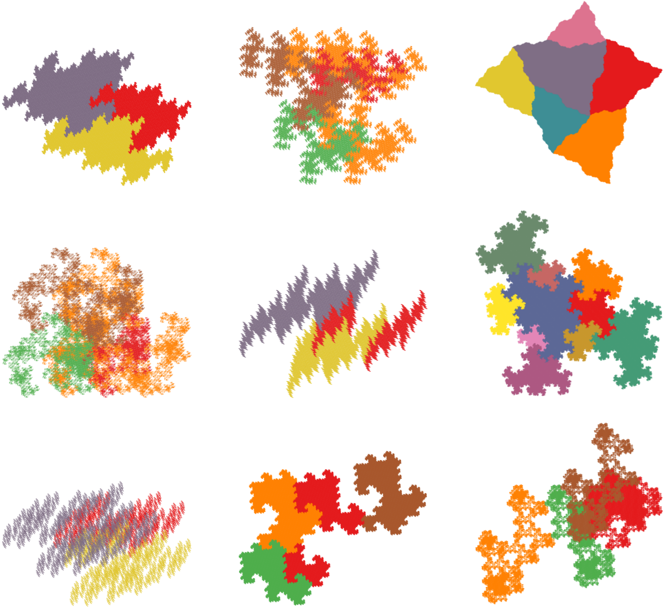


Rauzy fractals

Properties

- Compact set $\mathcal{T}$ s.t. $\mathcal{T} = \overline{\mathrm{int}(\mathcal{T})}$
- Self-affine
- Rich topology

Rich topology. . .



Dynamics of substitutions

- $\sigma : 1 \mapsto 12, \; 2 \mapsto 1312, \; 3 \mapsto 112$

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- $x = 1$

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- $X_\sigma = \overline{\{\text{shift}^n(x) : n \in \mathbb{Z}\}}$

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- Symbolic dynamcal system $(X_\sigma, \text{shift})\ldots$

Dynamics of substitutions

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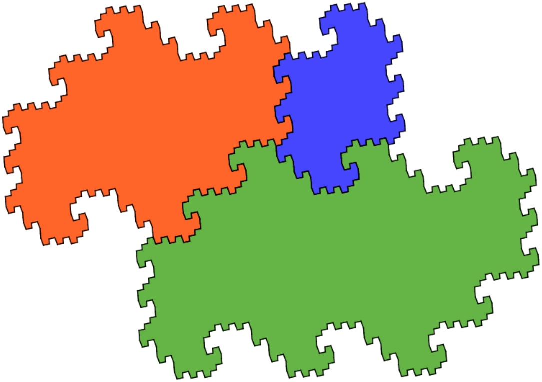
- $X_\sigma = \overline{\{\mathsf{shift}^n(x) : n \in \mathbb{Z}\}}$
- Symbolic dynamcal system $(X_\sigma, \mathsf{shift})$...
- Minimal, zero entropy, no periodic points...

Dynamics of substitutions



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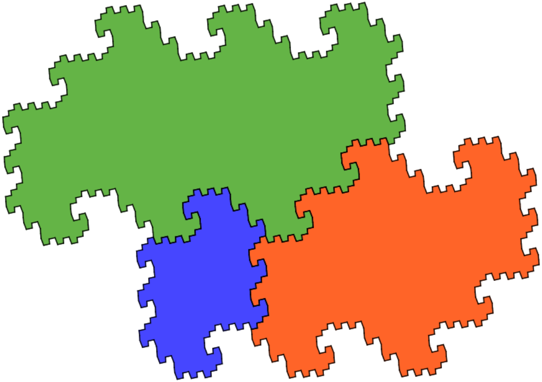

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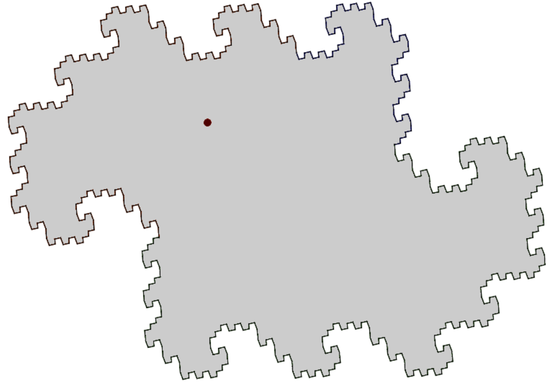

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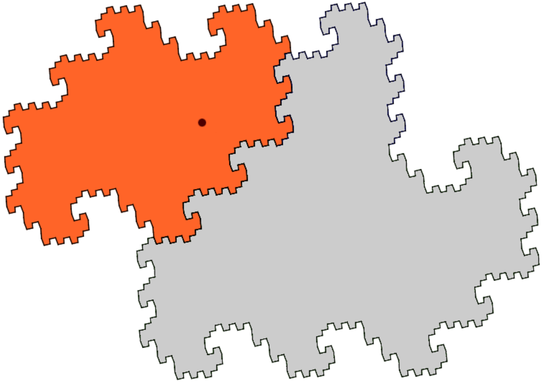

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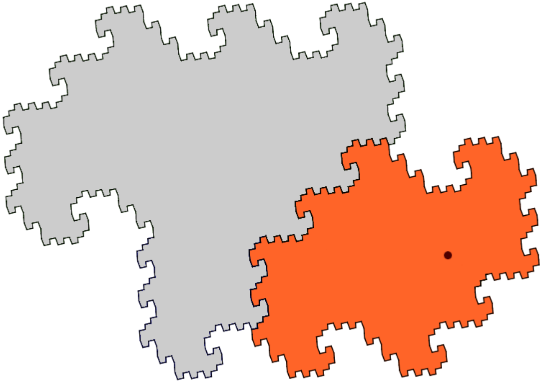

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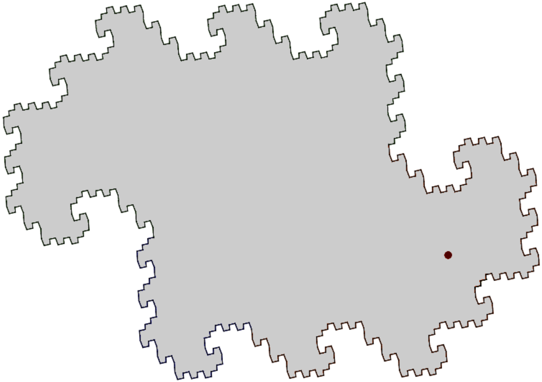
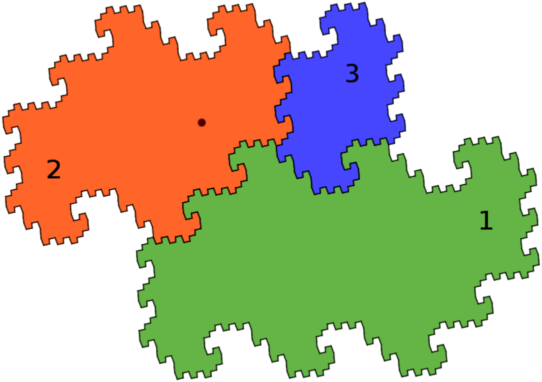


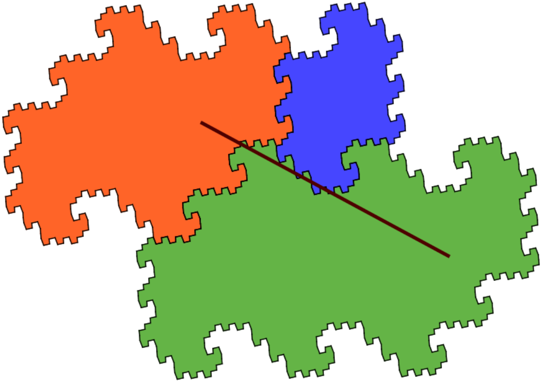
Dynamics of substitutions

Orbit : 2



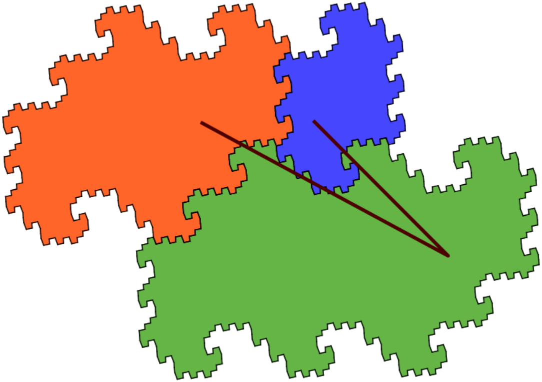
Dynamics of substitutions

Orbit : 21



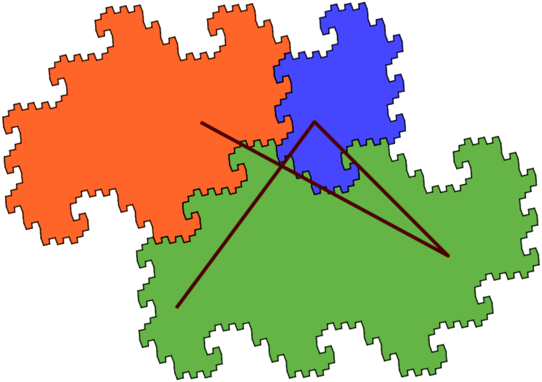
Dynamics of substitutions

Orbit : 213



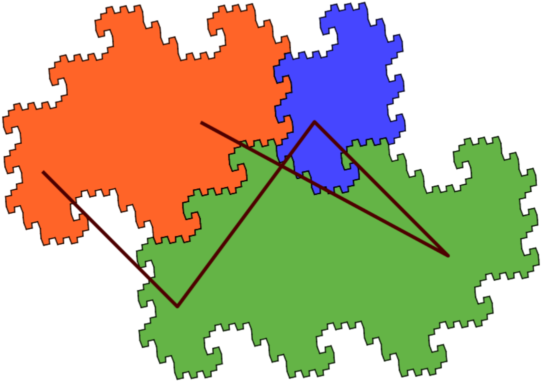
Dynamics of substitutions

Orbit : 2131



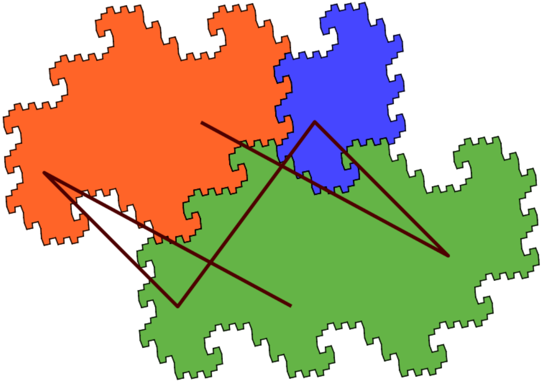
Dynamics of substitutions

Orbit : 21312



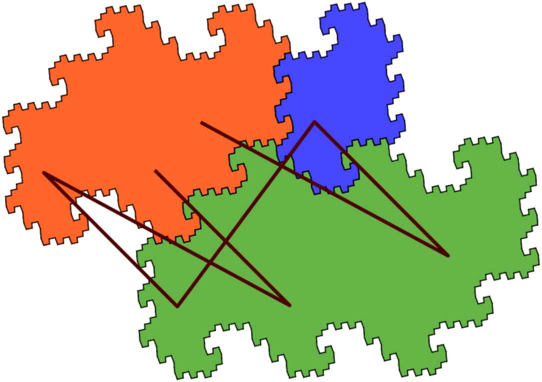
Dynamics of substitutions

Orbit : 213121



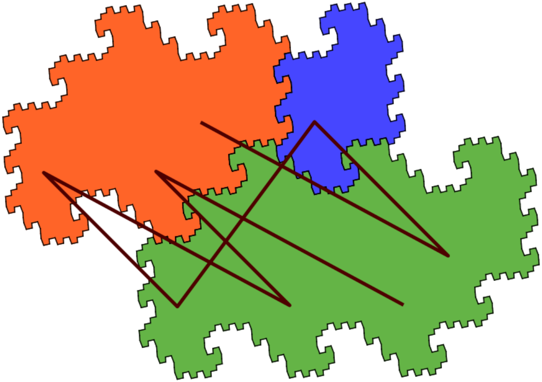
Dynamics of substitutions

Orbit : 2131212



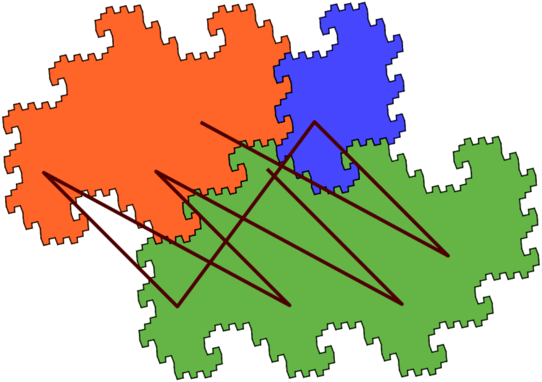
Dynamics of substitutions

Orbit : 21312121



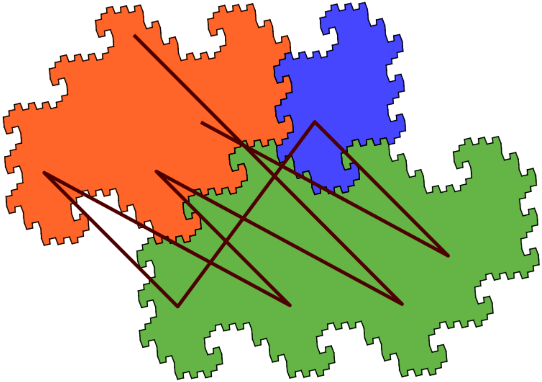
Dynamics of substitutions

Orbit : 213121211



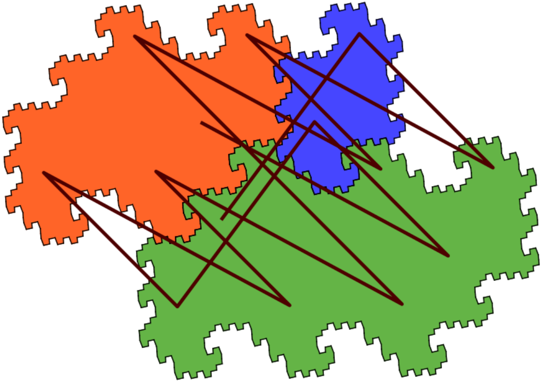
Dynamics of substitutions

Orbit : 2131212112



Dynamics of substitutions

Orbit : 2131212112121311



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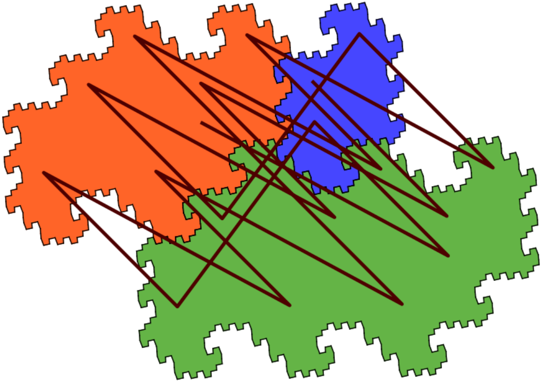
Orbit : 21312121121213121213



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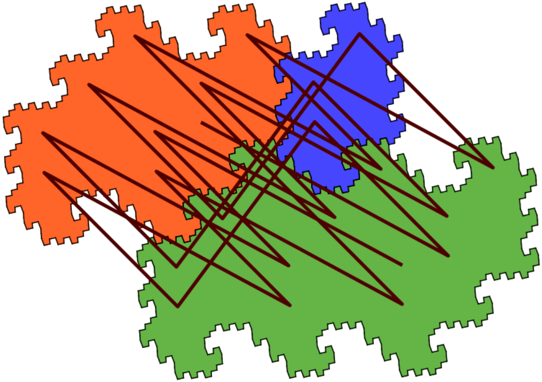

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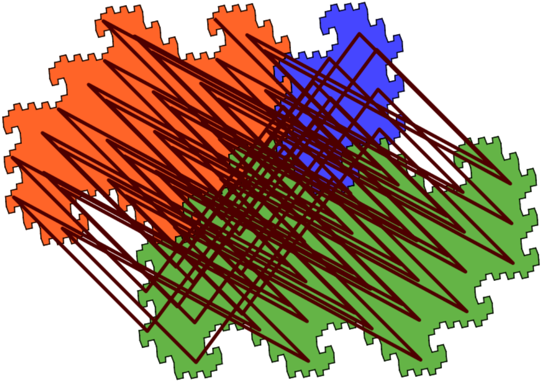
Orbit : $\cdots 213121211212131212131212 1\cdots \in X_\sigma \subseteq \{1,2,3\}^{\mathbb{Z}}$



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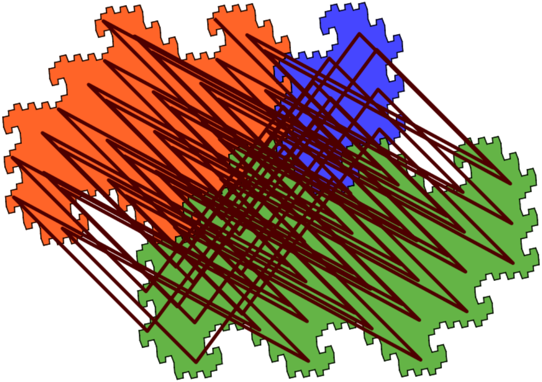
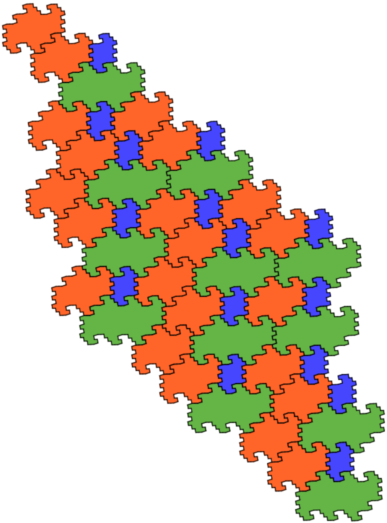
Orbit : $\cdots 2131212112121312121312121 \cdots \in X_\sigma \subseteq \{1,2,3\}^{\mathbb{Z}}$



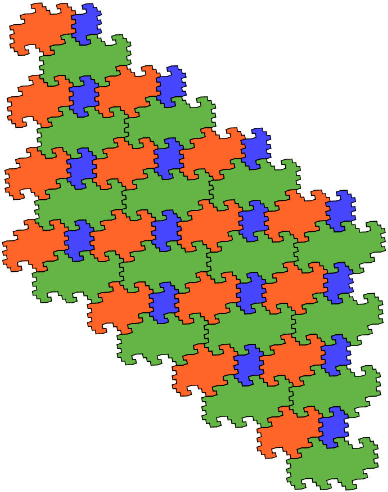
$\left(X_\sigma\right.$, shift$\left.\right) \quad \cong \quad \big($  , exchange $\big)$

Tilings

Self-similar tiling (aperiodic) :

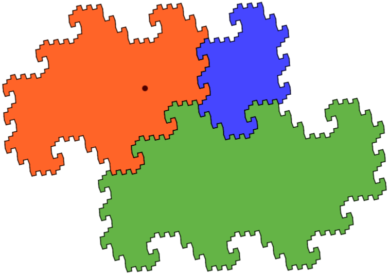


Periodic tiling :

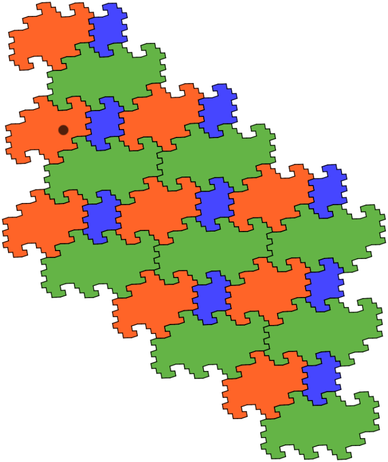


Dynamics of substitutions

(1) ( , exchange)



(2) $\left(\mathbb{T}^2, \text{translation}\right)$

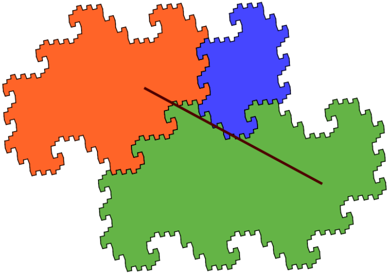


(3) $\left(X_\sigma, \text{shift}\right)$

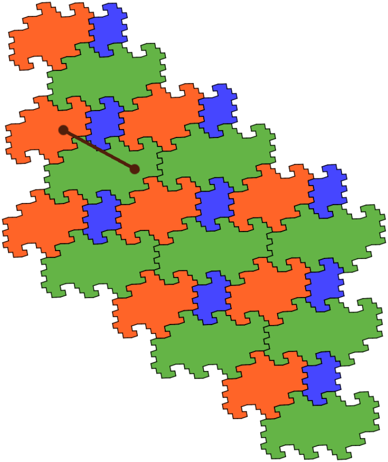
$\cdots \underline{2}131212112 \cdots \in X_\sigma$

Dynamics of substitutions

(1) ( , exchange)



(2) $\left(\mathbb{T}^2, \text{translation}\right)$

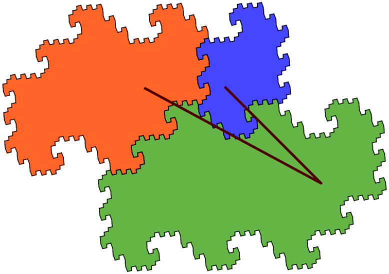


(3) $\left(X_\sigma, \text{shift}\right)$

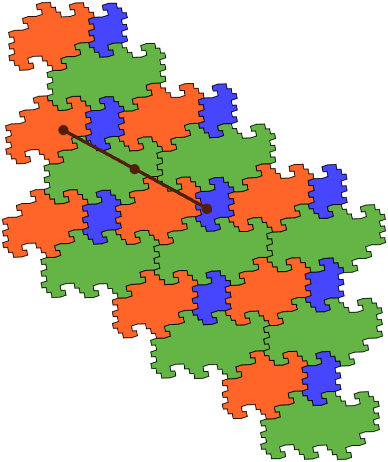
$\cdots 2\underline{1}31212112 \cdots \in X_\sigma$

Dynamics of substitutions

(1) $\Big($  , exchange $\Big)$



(2) $\Big(\mathbb{T}^2$, translation $\Big)$



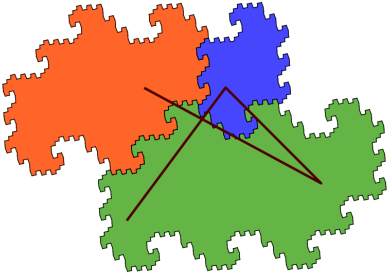
(3) $\Big(X_\sigma$, shift $\Big)$

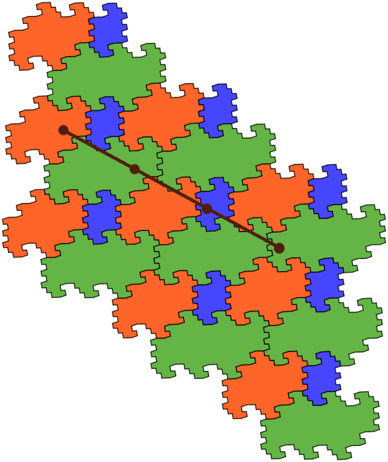
$\cdots 21\underline{3}1212112 \cdots \in X_\sigma$

Dynamics of substitutions

(1) ( , exchange)

(2) $\left(\mathbb{T}^2, \text{translation}\right)$



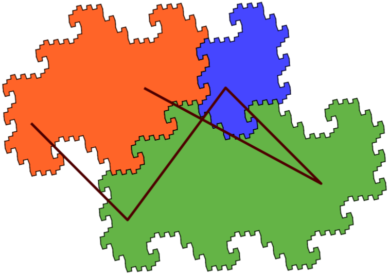


(3) $\left(X_\sigma, \text{shift}\right)$

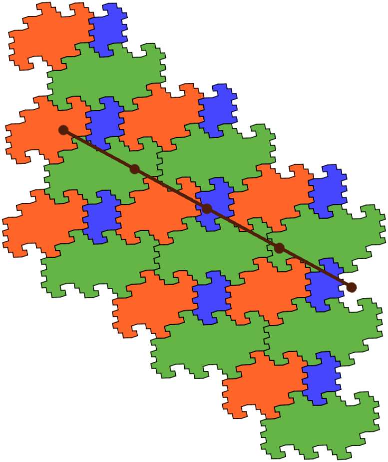
$\cdots 213\underline{1}212112 \cdots \in X_\sigma$

Dynamics of substitutions

(1) $\Big($  , exchange $\Big)$



(2) $\Big(\mathbb{T}^2$, translation $\Big)$



(3) $\Big(X_\sigma$, shift $\Big)$

$$\cdots 2131\underline{2}12112 \cdots \in X_\sigma$$

Dynamics of substitutions

(1) $\Big($  , exchange $\Big)$

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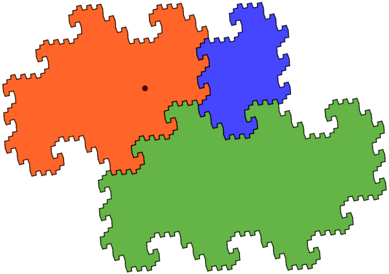

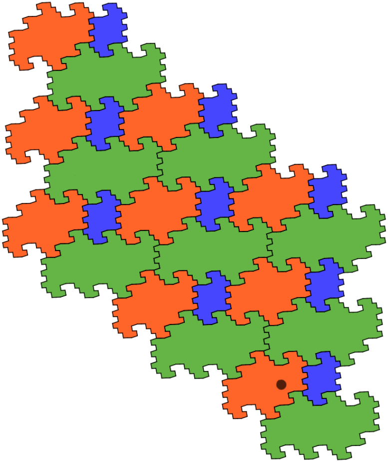

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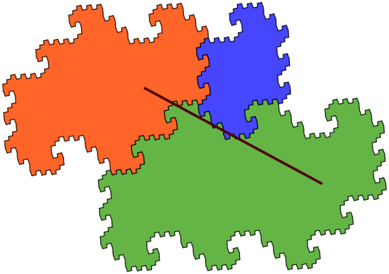

(3) $\Big(X_\sigma \text{, shift}\Big)$

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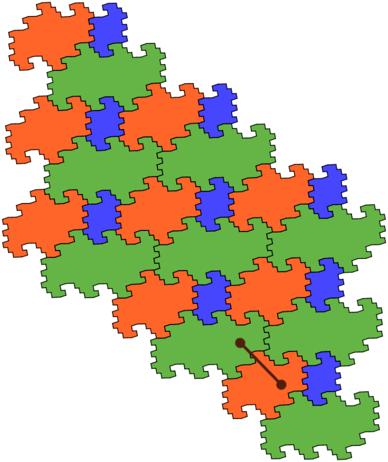
(2) $\Big(\mathbb{T}^2 \text{, translation}\Big)$



Dynamics of substitutions

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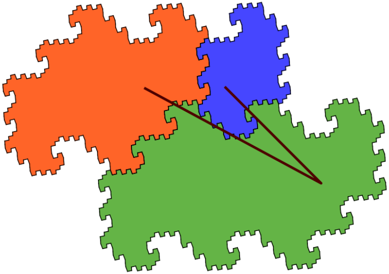

(3) $\Big(X_\sigma$, shift $\Big)$

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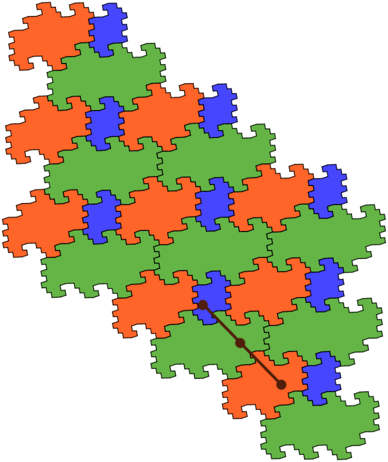

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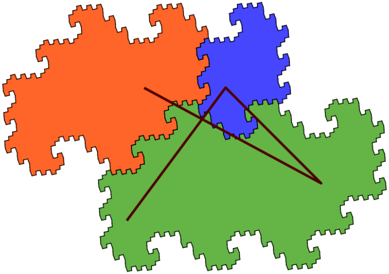
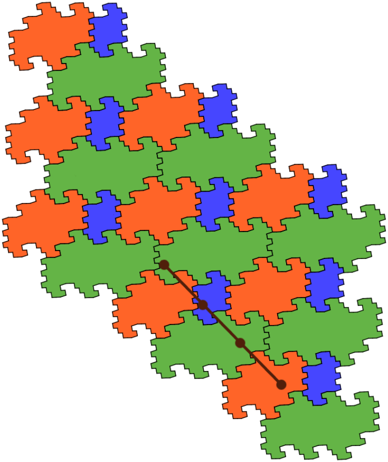
(2) $\Big(\mathbb{T}^2$, translation $\Big)$





(3) $\Big(X_\sigma$, shift $\Big)$

$\cdots 213\underline{1}212112 \cdots \in X_\sigma$

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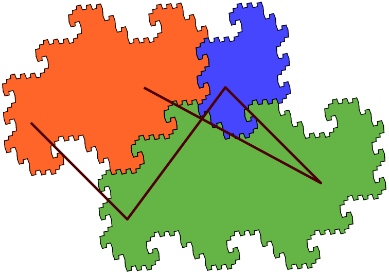

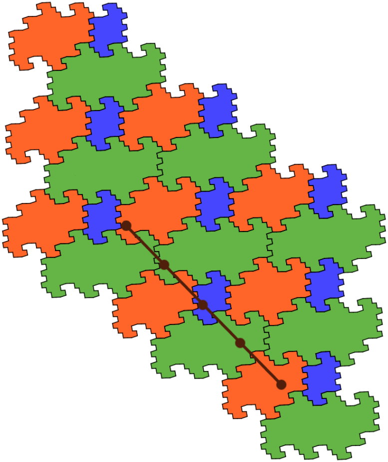

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Dynamics of substitutions

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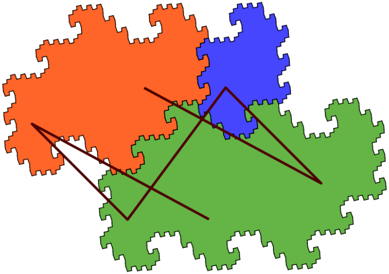
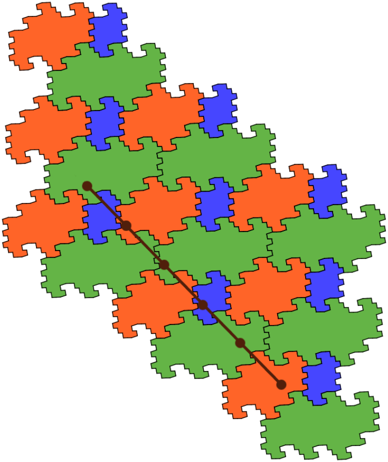


(3) $\Big(X_\sigma$, shift$\Big)$

$\cdots 21312\underline{1}2112 \cdots \in X_\sigma$

(2) $\Big(\mathbb{T}^2$, translation$\Big)$



Dynamics of substitutions

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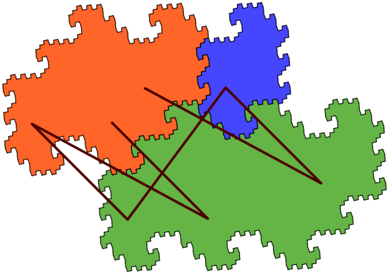
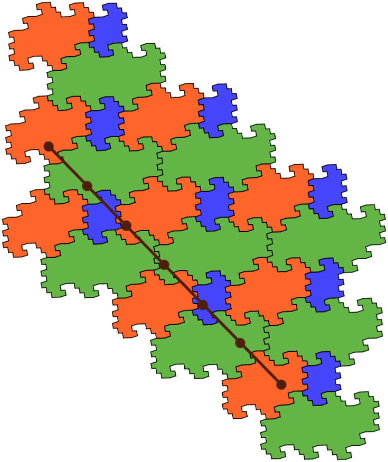


(2) $\Big(\mathbb{T}^2$, translation $\Big)$



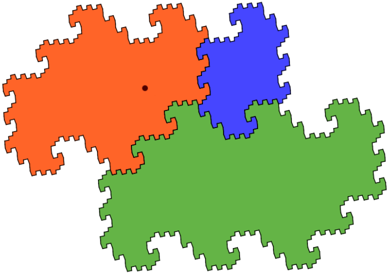
(3) $\Big(X_\sigma$, shift $\Big)$

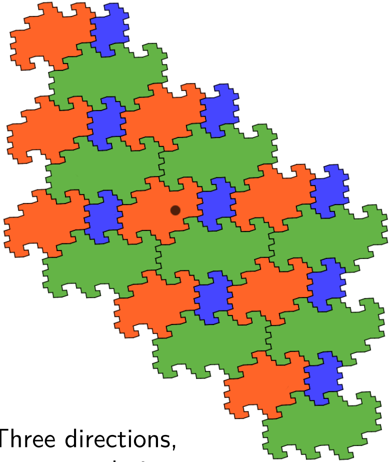
$\cdots 213121\underline{2}112 \cdots \in X_\sigma$

Dynamics of substitutions

(1) $\Big($  , exchange $\Big)$

(2) $\Big(\mathbb{T}^2$, translation $\Big)$





Three directions,
same translation

(3) $\Big(X_\sigma$, shift $\Big)$

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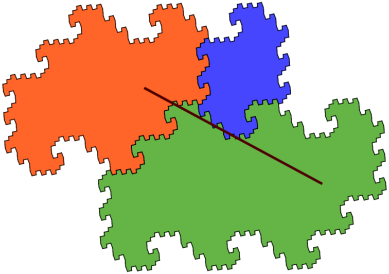
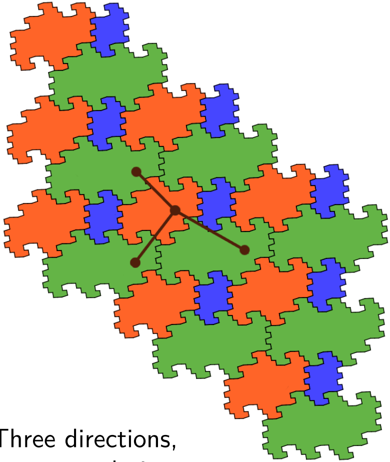
Dynamics of substitutions

(1) ( , exchange)



(2) ($\mathbb{T}^2$, translation)



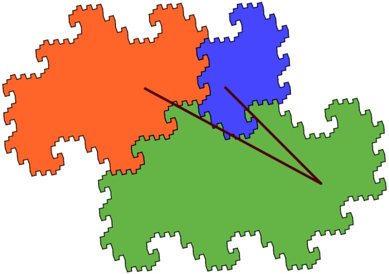
(3) (X_σ , shift)

$\cdots$ 2$\underline{1}$31212112 $\cdots \in X_\sigma$

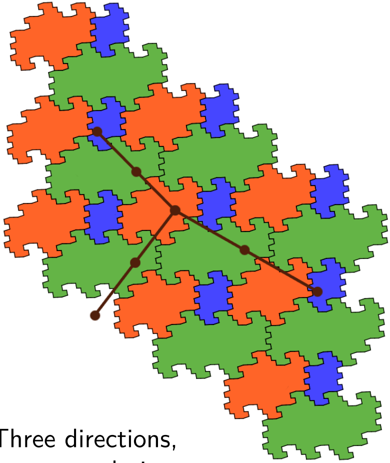
Three directions,
same translation

Dynamics of substitutions

(1) $\Big($  , exchange $\Big)$



(2) $\Big(\mathbb{T}^2$, translation $\Big)$



Three directions,
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(3) $\Big(X_\sigma$, shift $\Big)$

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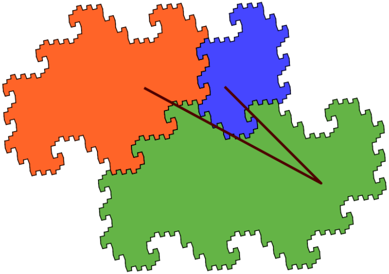
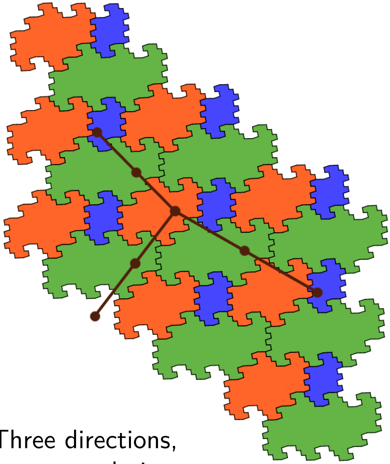
Dynamics of substitutions

(1) ( , exchange)

(2) $\left(\mathbb{T}^2, \text{translation}\right)$





(3) $\left(X_\sigma, \text{shift}\right)$

$\cdots 21\underline{3}1212112 \cdots \in X_\sigma$

Three directions, same translation

(1) $\cong$ **(2)** $\cong$ **(3)**

Properties of substitutions

Do we **always** have

- $\Big(X_\sigma\,,\text{shift}\Big) \;\cong\; \Big($  $,\text{exchange}\Big) \;\cong\; \Big(\mathbb{T}^2\,,\text{translation}\Big)$?

Properties of substitutions

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- Tiling properties?

➥ **Pisot conjecture**

Properties of substitutions

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➥ **Pisot conjecture**

Topological properties of $\mathcal{T}_\sigma$?

Properties of substitutions

Answers

- Easy to answer for **one given** substitution
  - Many algorithms, coincidence conditions
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  - Many algorithms, coincidence conditions
  - Many topological properties are decidable
- Open problem in general
- Some results for **infinite families**
  - $1 \mapsto 1^n 2, 2 \mapsto 1^m 3, 3 \mapsto 1$, for $n, m \geqslant 1$
  - β-numeration...
  - 2-letter case fully solved
  - **Today:** products of substitutions

Products of substitutions

Products of substitutions

Let $\sigma_1, \sigma_2, \sigma_3$ be substitutions.

What can we say about arbitrary products, like

$$\sigma_1\sigma_3,$$

$$\sigma_2^{40}\sigma_3\sigma_1^{1000}\sigma_3$$

or $\sigma_3(\sigma_3\sigma_2)^5\sigma_2\sigma_1\sigma_3$ **?**

Example: Brun substitutions

$\sigma_1 : 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 32$
$\sigma_2 : 1 \mapsto 1,\ 2 \mapsto 3,\ 3 \mapsto 23$
$\sigma_3 : 1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 13$

- $\sigma = \sigma_1\sigma_2\sigma_2$: **not Pisot!** We need at least one σ_3

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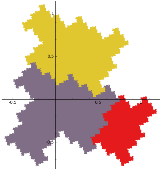
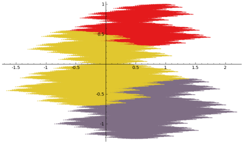
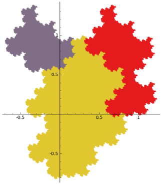


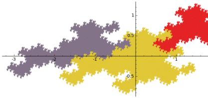
$\sigma = \sigma_1\sigma_3$



$\sigma = \sigma_2\sigma_3$



$\sigma = \sigma_3\sigma_1\sigma_2^2\sigma_3$



$\sigma = \sigma_2{\sigma_3^{10}}\sigma_2$

Example: Selmer substitutions

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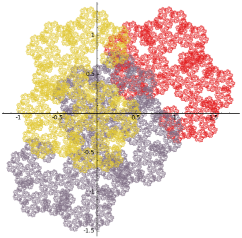
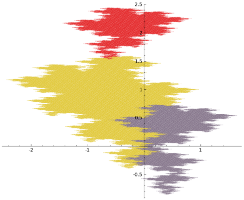
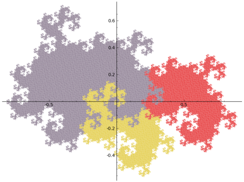
- We need at least one σ_3
- **Not Pisot:** $\sigma_1\sigma_3,\ \ \sigma_3\sigma_1\sigma_3\sigma_2,\ \ \sigma_1^3\sigma_2\sigma_3^2,\ \ \ldots$
- **Pisot:** $\sigma_3,\ \ \sigma_3^2\sigma_2^3,\ \ \sigma_3\sigma_1\sigma_3,\ \ \ldots$



σ_3



$\sigma_3^2\sigma_2^3$



$\sigma_3\sigma_1\sigma_3$

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- **Pisot:** $\sigma_3,\ \ \sigma_3^2\sigma_2^3,\ \ \sigma_3\sigma_1\sigma_3,\ \ \dots$

- Difficult **matrix** problems
- Difficult **word combinatorics** problems
- Unmanageable. . .

Products of substitutions

- Arbitrary products $\longrightarrow$ **no good algebraic representation**

Products of substitutions

- Arbitrary products $\longrightarrow$ **no good algebraic representation**
- We need other tools
  ➥ **dual substitutions**

Tools: dual substitutions

Arnoux and Ito introduced **dual substitutions** $\mathbf{E}_1^*(\sigma)$:

$$\sigma \quad \stackrel{\text{duality}}{\longmapsto} \quad \mathbf{E}_1^*(\sigma)$$

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$$\sigma \quad \xrightarrow{\text{duality}} \quad \mathbf{E}_1^*(\sigma)$$

$$\mathbf{E}_1^*(\sigma)([\mathbf{x}, i]^\star) = \bigcup_{(p,j,s)\in\mathcal{A}^\star\times\mathcal{A}\times\mathcal{A}^\star \;:\; \sigma(j)=pis} [\mathbf{M}_\sigma^{-1}(\mathbf{x}+\ell(s)), j]^\star$$

Tools: dual substitutions

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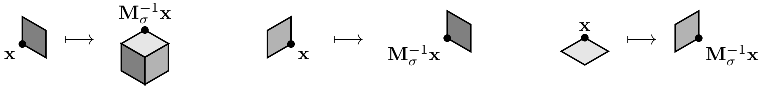
$$\sigma \quad \overset{\text{duality}}{\longmapsto} \quad \mathbf{E}_1^*(\sigma)$$

Example: $\mathbf{E}_1^*(\sigma)$ for $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$

$$\begin{array}{lcl} [\mathbf{x}, 1]^* & \mapsto & \mathbf{M}_\sigma^{-1}\mathbf{x} + [(1,0,-1),1]^* \cup [(0,1,-1),2]^* \cup [\mathbf{0},3]^* \\ [\mathbf{x}, 2]^* & \mapsto & \mathbf{M}_\sigma^{-1}\mathbf{x} + [\mathbf{0},1]^* \\ [\mathbf{x}, 3]^* & \mapsto & \mathbf{M}_\sigma^{-1}\mathbf{x} + [\mathbf{0},2]^* \end{array}$$

where $[\mathbf{x}, i]^*$ = face of type $i \in \{1,2,3\}$ placed at $\mathbf{x} \in \mathbb{Z}^3$



Tools: dual substitutions

Iterating $\mathbf{E}_1^*(\sigma)\ldots$

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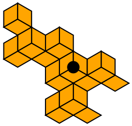

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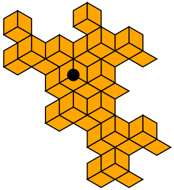
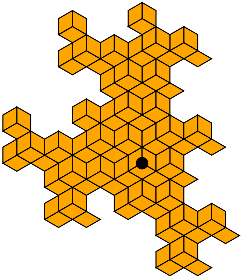


Tools: dual substitutions

Iterating $\mathbf{E}_1^*(\sigma)\ldots$



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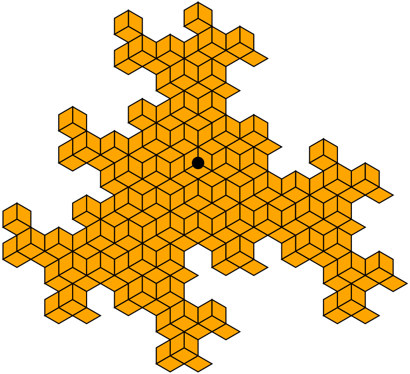

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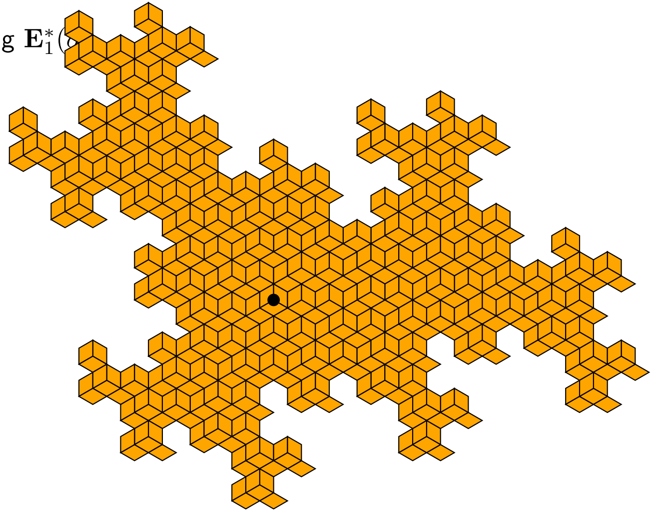
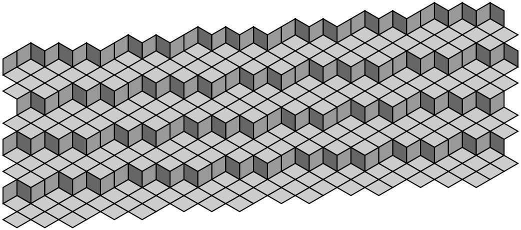


Striking connexion: $\mathrm{E}_1^*(\sigma)$ and discrete planes

Definition

Discrete plane $\Gamma_{\mathbf{v}} = \{[\mathbf{x}, i]^* : 0 \leqslant \langle \mathbf{x}, \mathbf{v} \rangle < \langle \mathbf{e}_i, \mathbf{v} \rangle\}$.

$\Gamma_{(1,\sqrt{2},\sqrt{17})}$:



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Proposition [Arnoux-Ito, Fernique]

$\mathbf{E}_1^*(\sigma)$ maps $\Gamma_{\mathbf{v}}$ to $\Gamma_{{}^t\mathbf{M}_\sigma \mathbf{v}}$

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The patches $\mathbf{E}_1^*(\sigma)^n($$)$ grow within discrete planes

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The patches $\mathbf{E}_1^*(\sigma)^n$() grow within discrete planes

Main question, leading to main tools:

How do these patches grow?

1. Do they cover arbitrarily large balls?
2. Do they cover arbitrarily large balls centered at $\mathbf{0}$?

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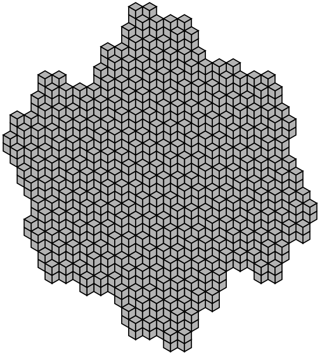
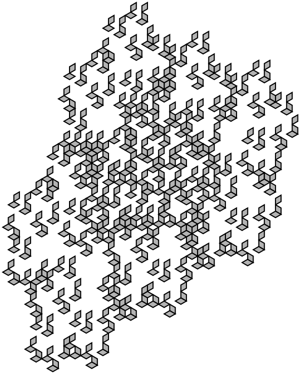
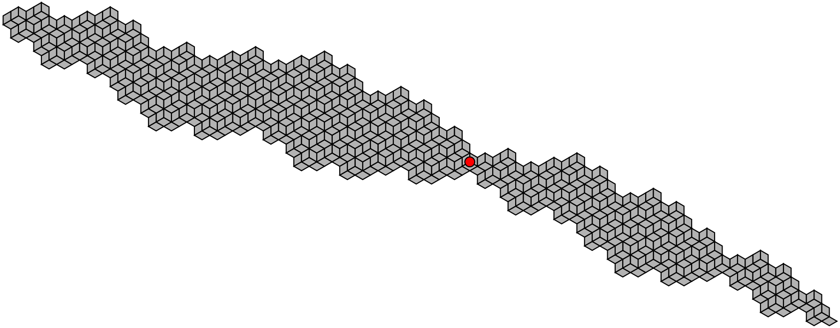




- Not always obvious...
- **But:** this is equivalent to the Pisot conjecture!

2. Do they cover arbitrarily large balls **centered at 0?**

Not always:



- Links with fractal topology (zero inner point)
- Links with number theory (finiteness properties)

Back to our original problem

What properties do **products** of $\sigma_1, \sigma_2, \sigma_3$ have?
(w.r.t. Pisot conjecture, topological properties)

is reduced to

Do the patches $\mathbf{E}_1^*(\sigma_{i_1}) \cdots \mathbf{E}_1^*(\sigma_{i_n})($$)$

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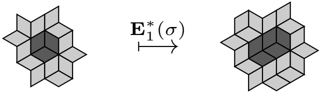
We will use combinatorial-geometrical methods to

- **Prove** 1
- **Characterize** the products satisfying 2

1. Proving that balls grow: annulus property



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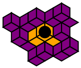

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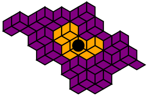

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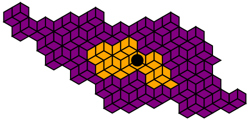

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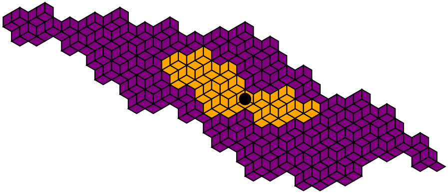

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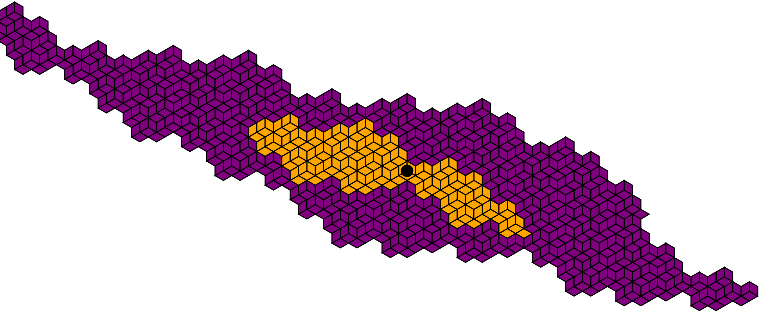

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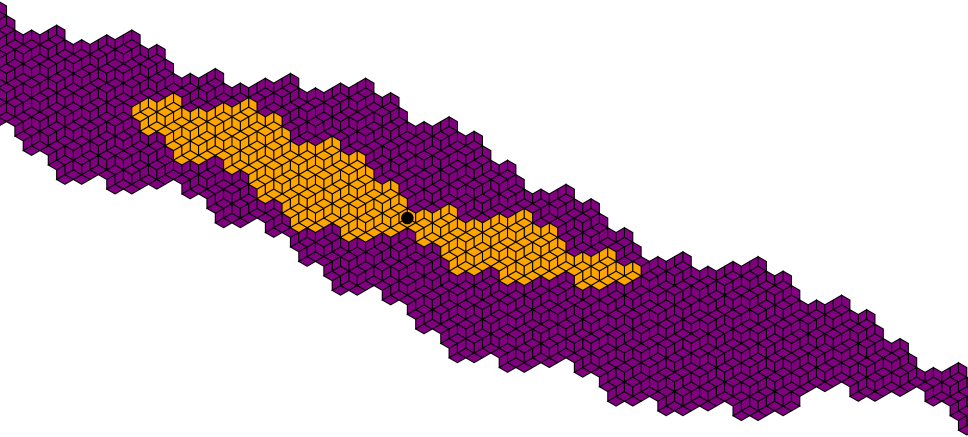


1. Proving that balls grow: annulus property

- Idea dates back to Ito-Ohtsuki 93
- Needs heavy combinatorics to make it work

2. Generation graphs

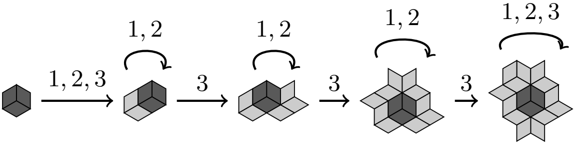
Two cases occur:

- **The seed  is enough to generate a whole plane**
- **The seed  is not enough**
  - Does there exist a finite seed, enough for all the products? (**yes** for a single σ, computable)

2. Generation graphs

Example where the seed  is enough:

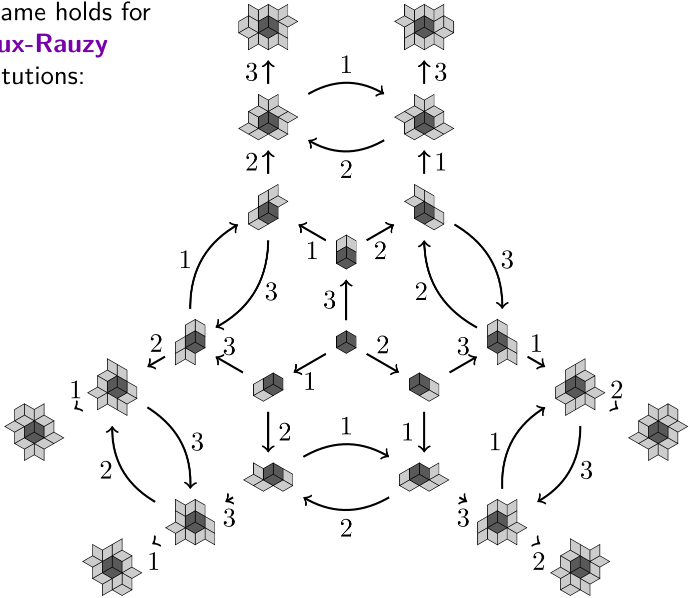
σ_1, σ_2, σ_3 associated with the **fully subtractive algorithm**
Infinite family: products containing at least one σ_3.



- Iterating the σ_i eventually yields an annulus.
- By induction: there are balls of arbitrarily large radius centered at $\mathbf{0}$.

2. Generation graphs

The same holds for **Arnoux-Rauzy** substitutions:



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Example where the seed is NOT enough: Brun substitutions

$(\mathbf{E}_1^*(\sigma_2)\mathbf{E}_1^*(\sigma_3))^\infty(\ )$

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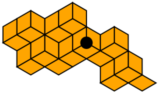

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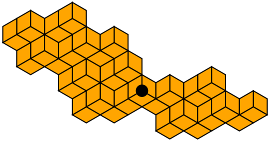

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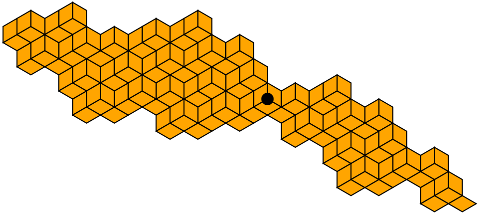

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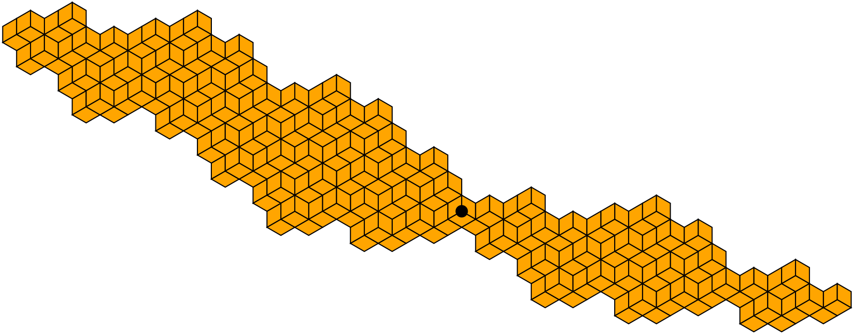
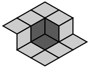
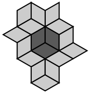


2. Generation graphs

Example where the seed  **is NOT enough**: **Brun substitutions**

- What is the **language** of the sequences $(i_n) \in \{1, 2, 3\}^{\mathbb{N}}$ for which  is enough?
- Can we compute a **seed** valid for every sequence (i_n)?

Main result

Let $\mathcal{V}_1 =$  and $\mathcal{V}_2 =$ 

Theorem [Berthé-Bourdon-J-Siegel]

Let $\sigma = \sigma_{i_1} \cdots \sigma_{i_n}$ be a product of Brun substitutions. We have:

1. $\mathcal{V}_1$ and $\mathcal{V}_2$ are good seeds for σ
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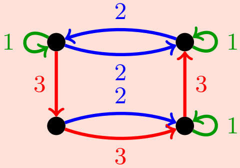

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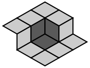
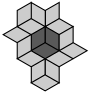
Let $\mathcal{V}_1 =$  and $\mathcal{V}_2 =$ 

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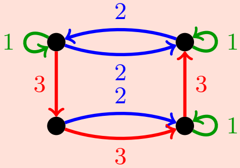

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➥ Algorithmic construction of the graph.

➥ Same results for Jacobi-Perron [Ito-Ohtsuki], Arnoux-Rauzy [Berthé-J-Siegel], Modified JP [Furukado-Ito-Yasutomi], ...

Applications: Dynamics

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- **Hence:**

Pisot conjecture holds for products of substitutions of Brun, Arnoux-Rauzy, Jacobi-Perron, . . .

Applications: topology of Rauzy fractals

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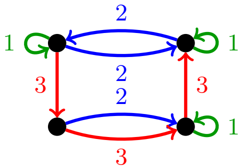

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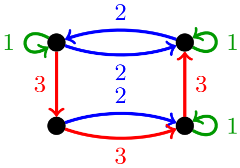

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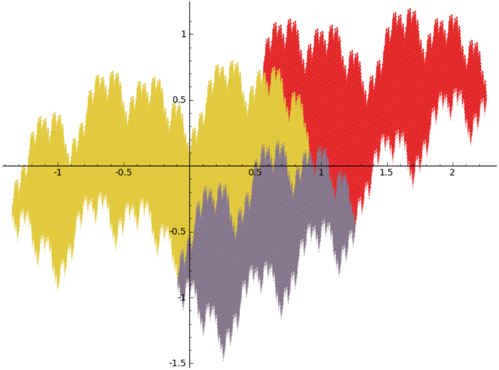

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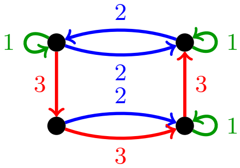

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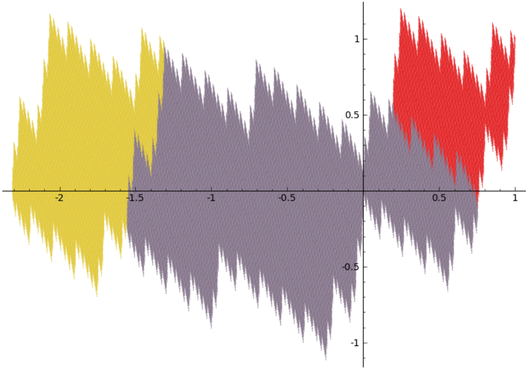

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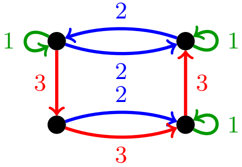

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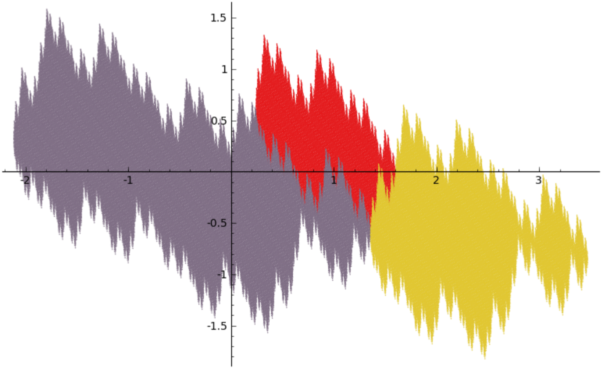

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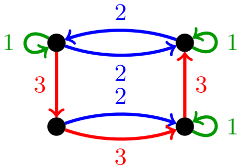

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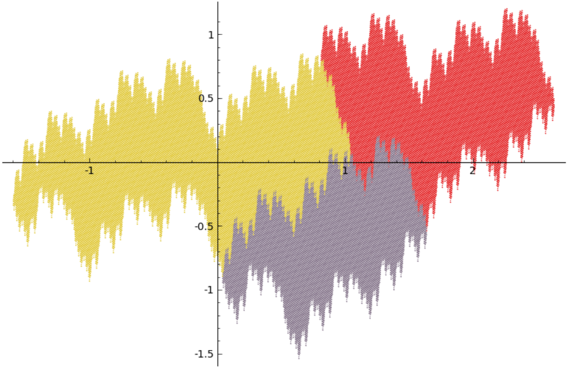

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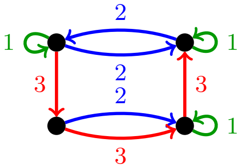

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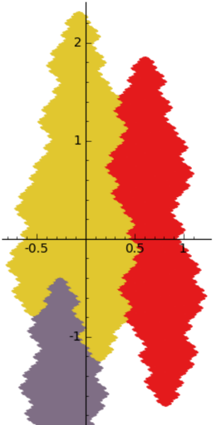

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Other topological properties:

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- The Rauzy fractals are connected
- **Interesting question:** which products yield simply connected fractals?

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Question

Computer experiments suggest:
The Pisot eigenvalue of $\mathbf{M}_{i_1} \cdots \mathbf{M}_{i_n}$ is **totally real**
when $i_1 \cdots i_n$ is in the language.

Why?

Conclusion and perspectives

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Thank you for your attention