Undecidable properties of self-affine sets and multi-tape automata

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Joint work with **Jarkko Kari**

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- **Self-affine sets:** “fractals” (object of study)
- **Multi-tape automata:** computational point of view (tool)

Example 1

- $f_1 : x \mapsto \frac{1}{2}x$
- $f_2 : x \mapsto \frac{1}{2}x + \binom{1/2}{0}$
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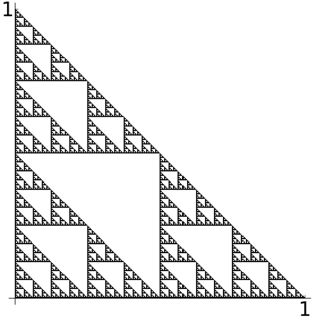

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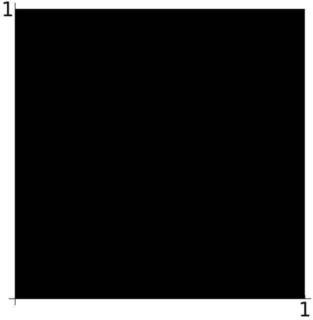

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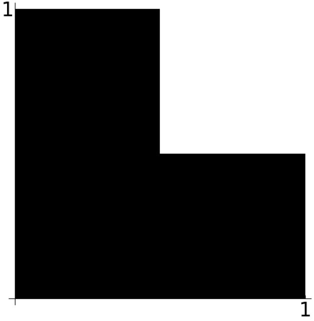

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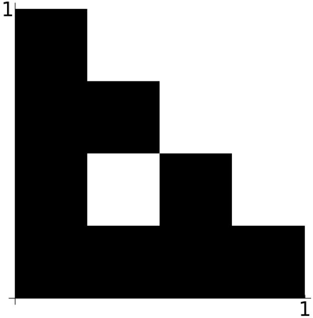

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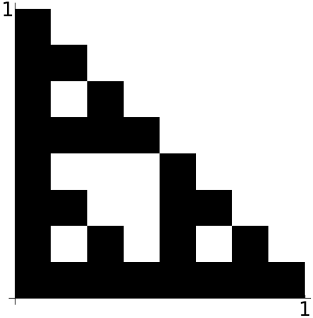

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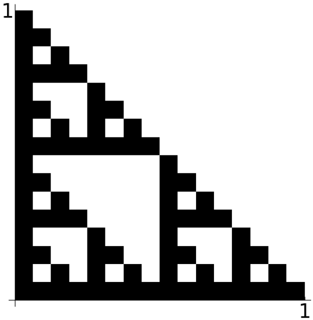

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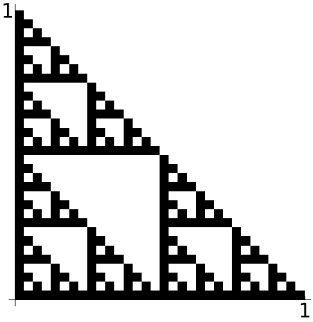

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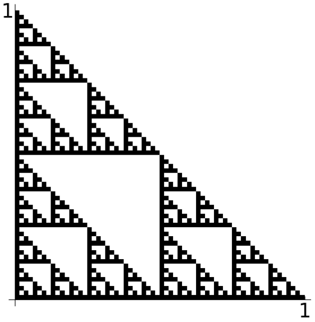

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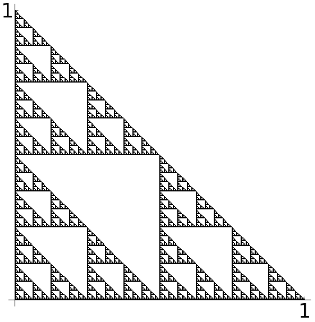

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Theorem (Hutchinson 1981)

Let $f_1, \ldots, f_n : \mathbb{R}^d \to \mathbb{R}^d$ be contracting maps.
There is a unique nonemtpy compact set $X \subseteq \mathbb{R}^d$ such that

$$X \; = \; f_1(X) \; \cup \; \cdots \; \cup \; f_n(X).$$

Definition: $f_1, \ldots, f_n$ is an **iterated function system (IFS)**

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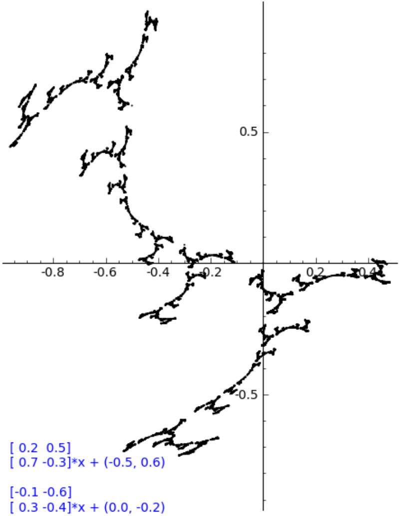

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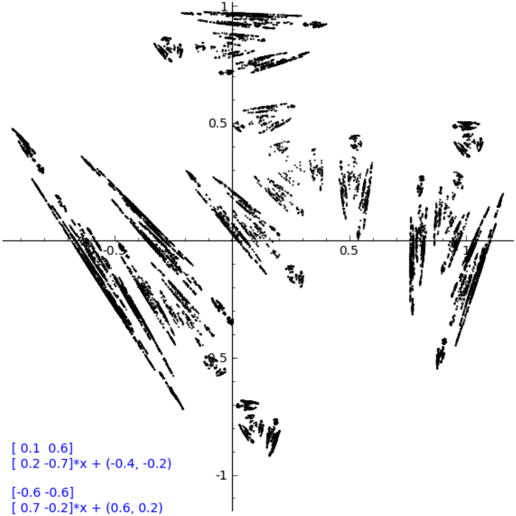

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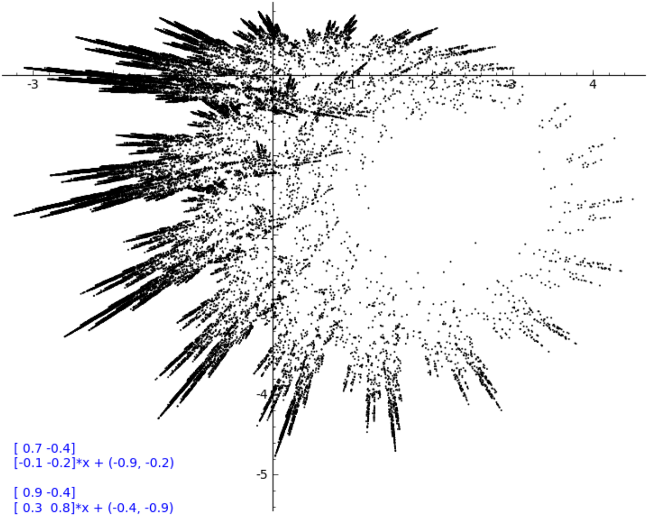

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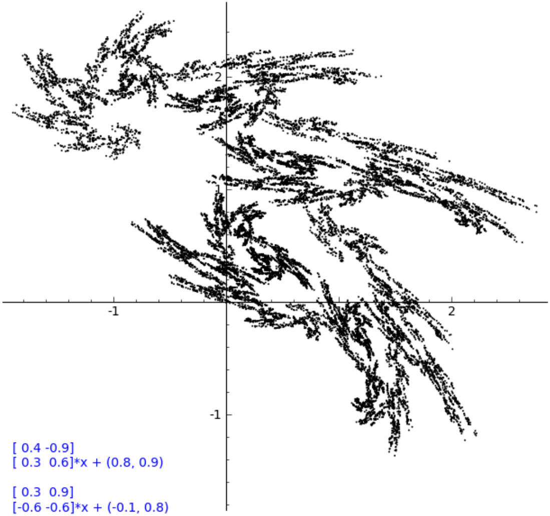

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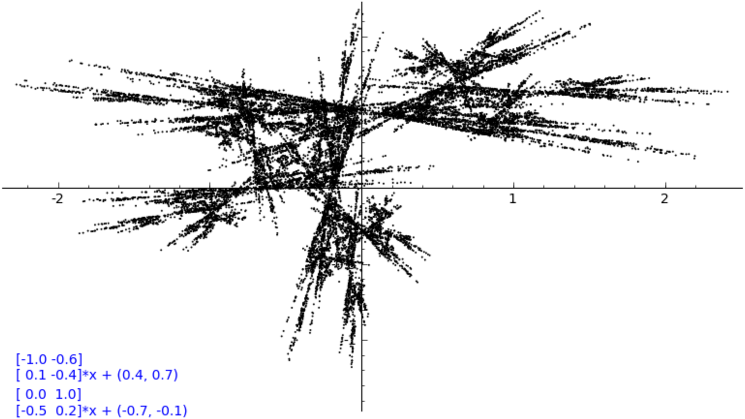

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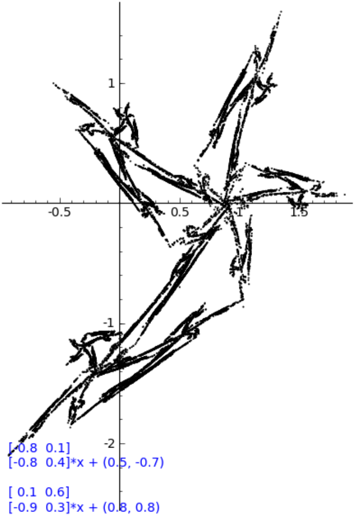

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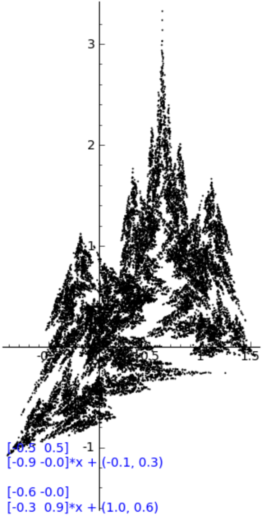

Affine iterated function systems

- Restrict to affine maps: $f_i : x \mapsto M_i x + v_i$
- Interesting topological objects:
  - Connectedness, homeomorphism to a disk, ...
  - Nonempty interior
  - Tiling properties
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We give "negative" answers to the nonempty interior question.

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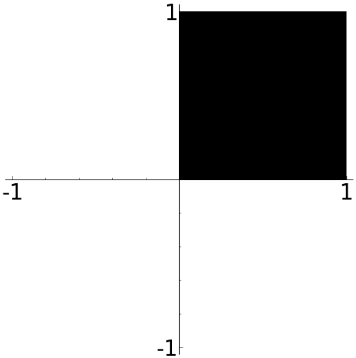

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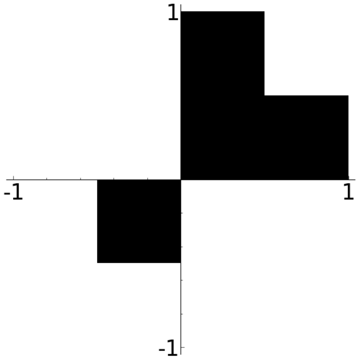

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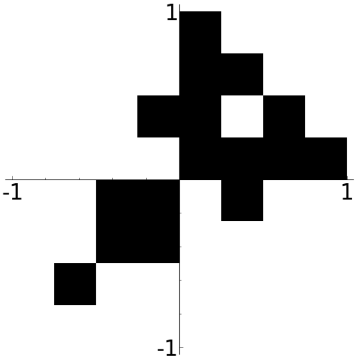

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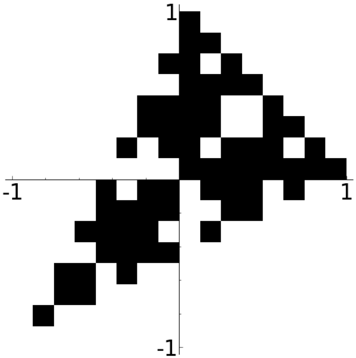

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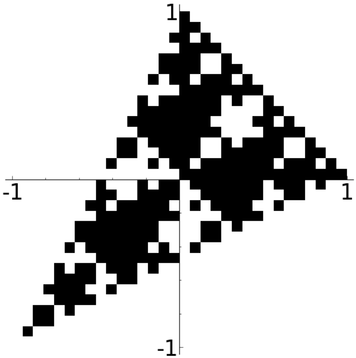

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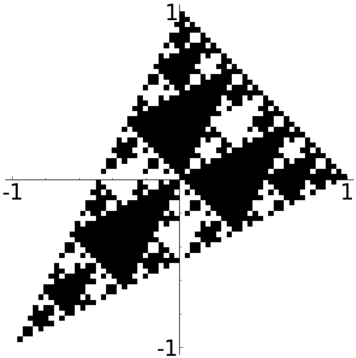

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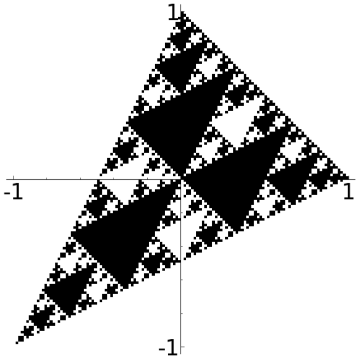

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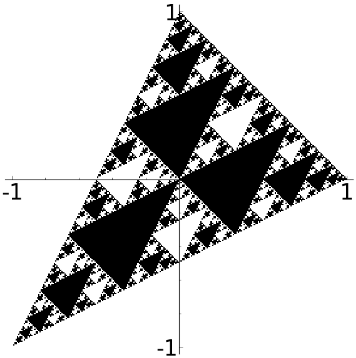

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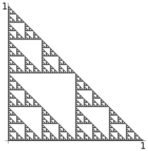
Example 1: 

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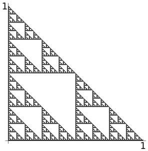
Example 1: 

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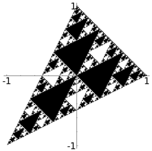
Example 2: 

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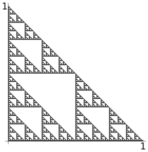
Example 1: 

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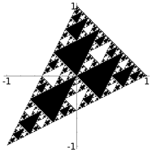
Example 2: 

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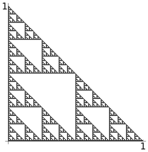
Example 1: 

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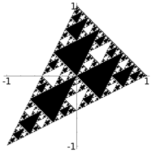
Example 2: 

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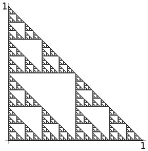
Example 1: 

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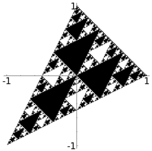
Example 2: 

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- There is an **algorithm** [Bondarenko-Kravchenko 2011] if the affine f_i all have the same contraction matrix (and verify a few other conditions)
- Almost nothing is known otherwise (even particular instances are difficult)
- Let's now go towards undecidability

Graph-IFS (GIFS)

$$f_1 : x \mapsto \frac{1}{2}x$$
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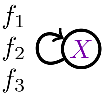

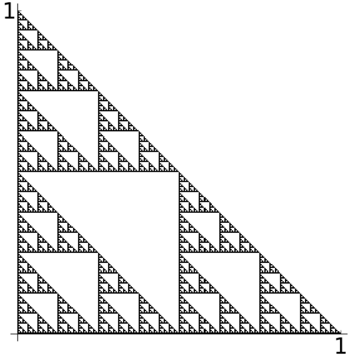

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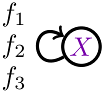
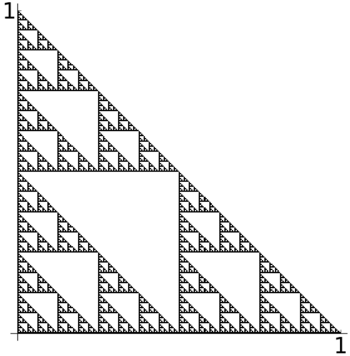
$$f_3 : x \mapsto \frac{1}{2}x + \binom{0}{1/2} \qquad f_6 : x \mapsto \frac{1}{2}x + \binom{3/4}{3/4}$$

$$X = f_1(X) \cup f_2(X) \cup f_3(X)$$





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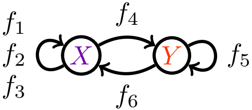
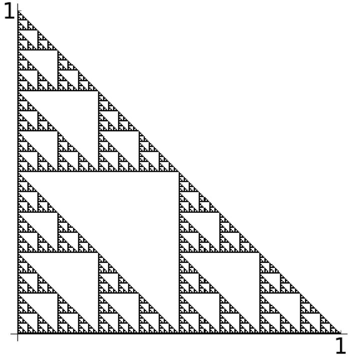
$$\begin{cases} X = f_1(X) \cup f_2(X) \cup f_3(X) \cup f_4(Y) \\ Y = f_5(Y) \cup f_6(X) \end{cases}$$





Graph-IFS (GIFS)

$$f_1 : x \mapsto \tfrac{1}{2}x \qquad f_4 : x \mapsto \tfrac{1}{2}x + \binom{1/2}{1/2}$$
$$f_2 : x \mapsto \tfrac{1}{2}x + \binom{1/2}{0} \qquad f_5 : x \mapsto \tfrac{1}{2}x + \binom{1/2}{3/4}$$
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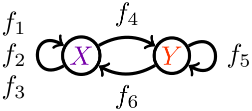

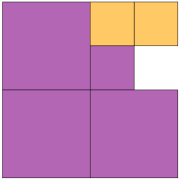

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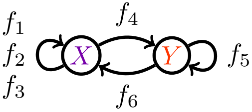

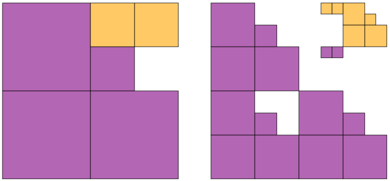

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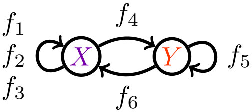

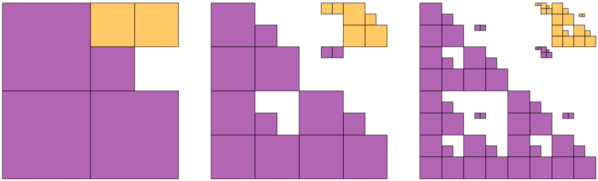

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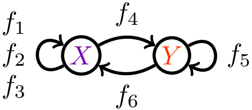

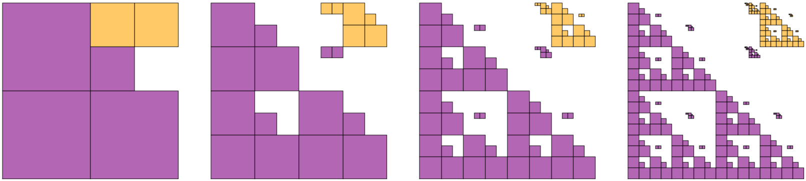

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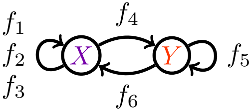
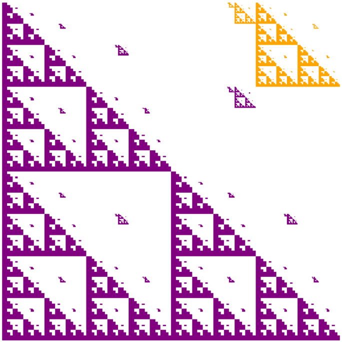


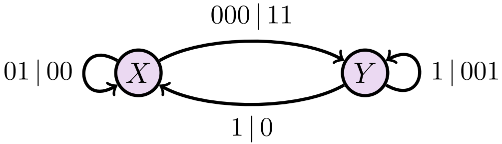


Computational tools: multi-tape automata

Multi-tape automata

d-tape automaton:

- alphabet $\mathcal{A} = A_1 \times \cdots \times A_d$
- states $\mathcal{Q}$
- transitions $\mathcal{Q} \times (A_1^+ \times \cdots \times A_d^+) \to \mathcal{Q}$



$\mathcal{A} = \{0,1\} \times \{0,1\}$
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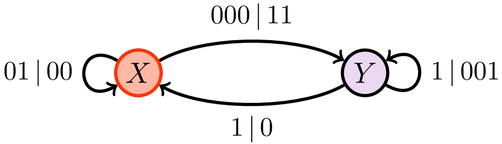

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Accepted infinite word starting from X:

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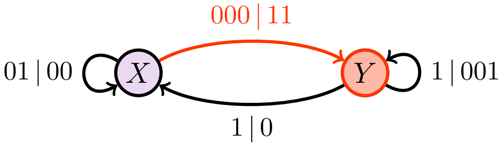

$\mathcal{A} = \{0,1\} \times \{0,1\}$
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Accepted infinite word starting from X:

$$\begin{matrix} 000 \\ 11 \end{matrix}$$

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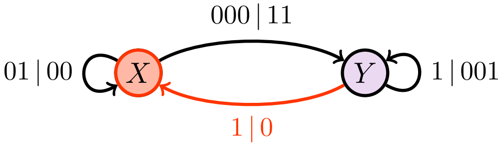

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Accepted infinite word starting from X:

0001
110

Multi-tape automata

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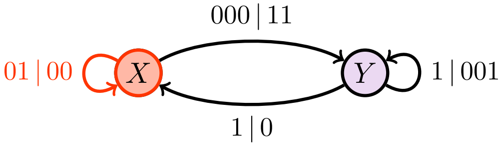

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000101
11000

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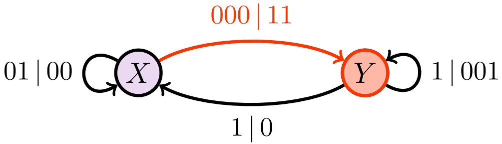

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000101000
1100011

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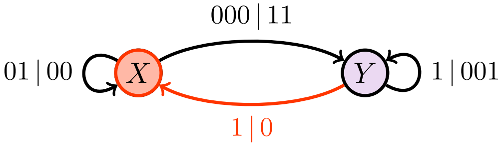

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Accepted infinite word starting from X:

0001010001
11000110

Multi-tape automata

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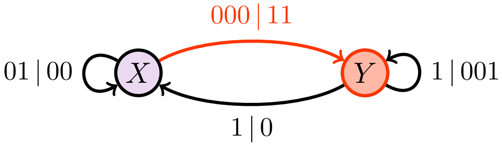

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0001010001000
1100011011

Multi-tape automata

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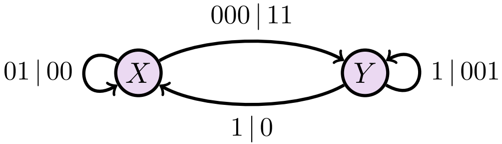

$\mathcal{A} = \{0,1\} \times \{0,1\}$
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Accepted infinite word starting from X:

$$\begin{matrix} 0001010001000\ldots \\ 1100011011\ldots \end{matrix} \in \mathcal{A}^{\mathbb{N}} = (\{0,1\} \times \{0,1\})^{\mathbb{N}}$$

Multi-tape automaton $\longmapsto$ GIFS

Transition 

(tape alphabets A_1, A_2)

Multi-tape automaton $\longmapsto$ GIFS

Transition 

(tape alphabets A_1, A_2)

$\longmapsto$ Mapping $$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} |A_1|^{-|u|} & 0 \\ 0 & |A_2|^{-|v|} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.u_1 \dots u_{|u|} \\ 0.v_1 \dots v_{|v|} \end{pmatrix}$$

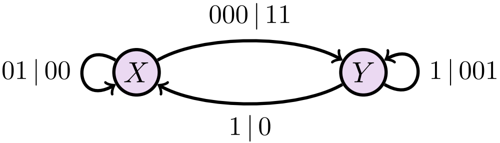
Multi-tape automaton $\longmapsto$ GIFS

Transition 

(tape alphabets A_1, A_2)

$\longmapsto$ Mapping $f\binom{x}{y} = \begin{pmatrix} |A_1|^{-|u|} & 0 \\ 0 & |A_2|^{-|v|} \end{pmatrix} \binom{x}{y} + \begin{pmatrix} 0.u_1 \dots u_{|u|} \\ 0.v_1 \dots v_{|v|} \end{pmatrix}$

Automaton:



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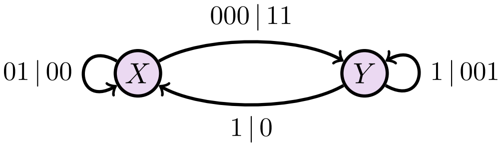
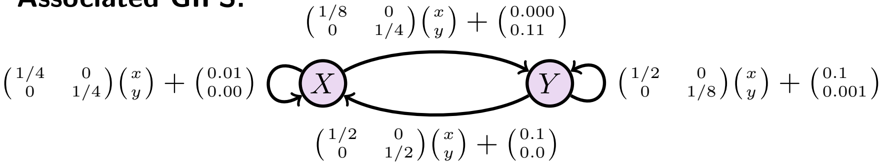


Associated GIFS:

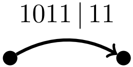


Multi-tape automaton $\longmapsto$ GIFS

Dictionary:

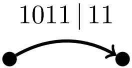
Automaton	GIFS fractal
states	GIFS fractals
edges	GIFS mappings
#tapes	dimension of fractals
alphabet A_i	base-$\lvert A_i \rvert$ representation of ith coordinate

Multi-tape automaton $\longmapsto$ GIFS

- 

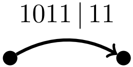
- $f\binom{x}{y} \quad = \quad \begin{pmatrix} 1/16 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0.x_1x_2\ldots \\ 0.y_1y_2\ldots \end{pmatrix} + \begin{pmatrix} 0.1011 \\ 0.11 \end{pmatrix}$

Multi-tape automaton $\longmapsto$ GIFS

- 

- $$f\binom{x}{y} = \begin{pmatrix} 1/16 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0.x_1x_2\dots \\ 0.y_1y_2\dots \end{pmatrix} + \begin{pmatrix} 0.1011 \\ 0.11 \end{pmatrix}$$
$$= \begin{pmatrix} 0.0000x_1x_2\dots \\ 0.00y_1y_2\dots \end{pmatrix} + \begin{pmatrix} 0.1011 \\ 0.11 \end{pmatrix}$$

Multi-tape automaton $\longmapsto$ GIFS

- 

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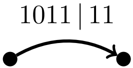
Multi-tape automaton $\longmapsto$ GIFS

- 

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$$= \begin{pmatrix} 0.1011x_1x_2\ldots \\ 0.11y_1y_2\ldots \end{pmatrix}$$

Key correspondence

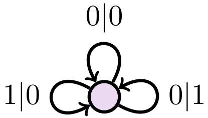
GIFS fractal associated with automaton $\mathcal{M}$

$$=$$

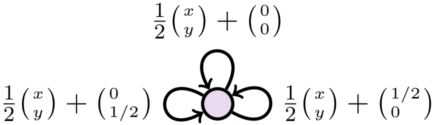
$$\left\{ \begin{pmatrix} 0.x_1x_2\ldots \\ 0.y_1y_2\ldots \end{pmatrix} \in \mathbb{R}^2 \ : \ \begin{pmatrix} x_1x_2\ldots \\ y_1y_2\ldots \end{pmatrix} \text{ accepted by } \mathcal{M} \right\}$$

Multi-tape automaton $\longmapsto$ GIFS

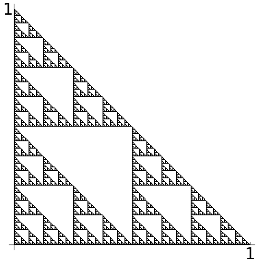
Example:



Automaton



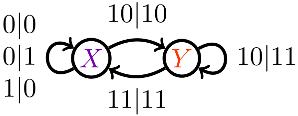
GIFS

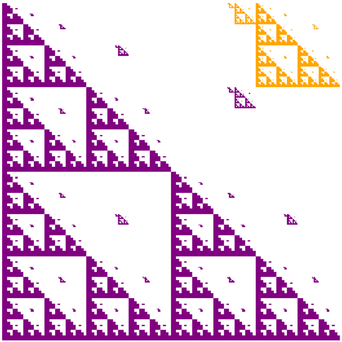


$$= \left\{ \begin{pmatrix} 0.x_1x_2\ldots \\ 0.y_1y_2\ldots \end{pmatrix} : (x_n, y_n) \neq (1,1),\ \forall n \geqslant 1 \right\}$$

Multi-tape automaton $\longmapsto$ GIFS

Example:





Multi-tape automaton $\longmapsto$ GIFS

Main idea

Language theoretical properties of accepted language

$\longleftrightarrow$

Topological properties of fractal set

Multi-tape automaton $\longmapsto$ GIFS

Main idea

Language theoretical properties of accepted language

$\longleftrightarrow$

Topological properties of fractal set

Example (in 2D) [Dube 1993, original idea]

Automaton accepts one word of the form $\binom{0.x_1x_2\dots}{0.x_1x_2\dots}$

$\Longleftrightarrow X$ intersects the diagonal $\{(x,x) : x \in [0,1]\}$

Language universality $\iff$ nonempty interior

Fact 1: $\mathcal{M}$ is universal $\iff$ $X = [0,1]^d$

Language universality $\iff$ nonempty interior

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- **Example (universal):** one state, transitions $0|0$, $0|1$, $1|0$, $1|1$

Language universality $\iff$ nonempty interior

Fact 1: $\mathcal{M}$ is universal $\iff X = [0,1]^d$

- **Example (universal):** one state, transitions $0|0$, $0|1$, $1|0$, $1|1$
- **Example (not universal):** one state, transitions $0|0$, $0|1$, $1|0$

Language universality $\iff$ nonempty interior

Fact 1: $\mathcal{M}$ is universal $\iff$ $X = [0,1]^d$

- **Example (universal):** one state, transitions $0|0$, $0|1$, $1|0$, $1|1$
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Fact 2: $\mathcal{M}$ is prefix-universal $\iff$ X has nonempty interior

Language universality $\iff$ nonempty interior

Fact 1: $\mathcal{M}$ is universal $\iff$ $X = [0,1]^d$

- **Example (universal):** one state, transitions $0|0$, $0|1$, $1|0$, $1|1$
- **Example (not universal):** one state, transitions $0|0$, $0|1$, $1|0$

Fact 2: $\mathcal{M}$ is prefix-universal $\iff$ X has nonempty interior

- **Example (universal with prefix 1 but not universal):** one state, transitions 1, 10, 00 (one-dimensional)

$f_1(x) = x/2 + 1/2$
$f_2(x) = x/4$
$f_3(x) = x/4 + 1/2$

Main result

Theorem [J-Kari 2013]

For 3-state, 2-tape automata:

- universality is undecidable
- prefix-universality is undecidable

Main result

Theorem [J-Kari 2013]

For 3-state, 2-tape automata:

- universality is undecidable
- prefix-universality is undecidable

Corollary

For 2D affine graph-IFS with 3 states:

- $= [0,1]^2$ is undecidable
- empty interior is undecidable

Proof idea

Post correspondence problem (undecidable)

Given n pairs of words $(u_1, v_1), \ldots, (u_n, v_n)$,
does there exists $i_1, \ldots, i_k$
such that $u_{i_1} \cdots u_{i_k} = v_{i_1} \cdots v_{i_k}$?

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Examples:

- There is a solution:
  $(u_1, v_1) = (aa, aab)$, $(u_2, v_2) = (bb, ba)$,
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  $(i_1, i_2, i_3, i_4) = (1, 2, 1, 3)$ is a solution because

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Variant: infinite-PCP

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Theorem [Dube 1993]

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  $\iff$ attractor $\cap$ diagonal $\neq \varnothing$
- **So:** "attractor $\cap$ diagonal $\neq \varnothing$" is undecidable

Proof idea

Theorem [J-Kari 2013]

For 3-state, 2-tape automata:

- universality is undecidable
- prefix-universality is undecidable

- Universality: reduce PCP
- Prefix-universality: reduce a variant of PCP (“prefix-PCP”)

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Property of the d-tape automaton	Topological property
$\exists$ configurations with $=$ tapes	Intersects the diagonal [Dube]
Is universal	Is equal to $[0,1]^d$
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