Generating discrete planes with substitutions

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With
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Classical case: lines coded in $\{1,2\}^{\mathbb{Z}}$

- Compute cont. frac. expansion of normal vector $\mathbf{v}$
- Iterate corresponding sequence of

$$\sigma_1 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \end{cases} \qquad \text{and} \qquad \sigma_2 : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \end{cases}$$

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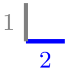
$\sigma_2(1) = 12$



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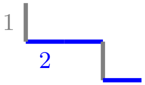
$\sigma_2\sigma_1\sigma_2(1) = 12212$



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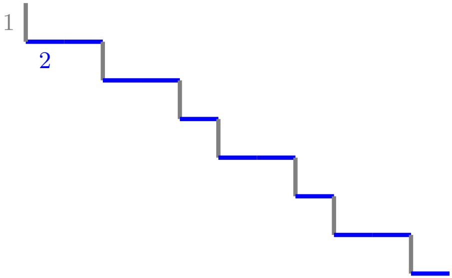

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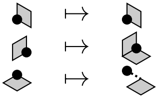
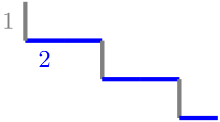
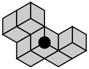
lines	**planes**
$\begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \end{cases}$	
	

Brun algorithm

- $\mathbf{v} \in \mathbb{R}^3$ such that $0 \leqslant \mathbf{v}_1 \leqslant \mathbf{v}_2 \leqslant \mathbf{v}_3$
- **Brun map:**

$$\mathbf{v} \;\mapsto\; \texttt{sort}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_2) \;=\; \begin{cases} (\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 - \mathbf{v}_2) & (1) \\ (\mathbf{v}_1,\ \mathbf{v}_3 - \mathbf{v}_2,\ \mathbf{v}_2) & (2) \\ (\mathbf{v}_3 - \mathbf{v}_2,\ \mathbf{v}_1,\ \mathbf{v}_2) & (3) \end{cases}$$

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- Iterate: **expansion** $(i_n) \in \{1, 2, 3\}^{\mathbb{N}}$

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  - $(1, e, \pi)$

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  - $\ldots$
  - **Expansion** $(i_n)_{n\in\mathbb{N}} = 3123233121111323 1123\ldots$

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Theorem [Brun 58]

1. $\mathbf{v}$ **totally irrational** $\iff$ $(i_n)_{n\in\mathbb{N}}$ contains **infinitely many** 3**'s**

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Theorem [Brun 58]

1. $\mathbf{v}$ **totally irrational** $\iff$ $(i_n)_{n\in\mathbb{N}}$ contains **infinitely many** 3**'s**
2. **Convergence:** to every such $(i_n)_{n\in\mathbb{N}}$ corresponds a **unique** $\mathbf{v}$

Brun substitutions

$$(\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 - \mathbf{v}_2) \quad (\mathbf{v}_1,\ \mathbf{v}_3 - \mathbf{v}_2,\ \mathbf{v}_2) \quad (\mathbf{v}_3 - \mathbf{v}_2,\ \mathbf{v}_1,\ \mathbf{v}_2)$$

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$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

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$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\sigma_i : \quad \begin{array}{l} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{array} \qquad \begin{array}{l} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 23 \end{array} \qquad \begin{array}{l} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 13 \end{array}$$

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	$(\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 - \mathbf{v}_2)$	$(\mathbf{v}_1,\ \mathbf{v}_3 - \mathbf{v}_2,\ \mathbf{v}_2)$	$(\mathbf{v}_3 - \mathbf{v}_2,\ \mathbf{v}_1,\ \mathbf{v}_2)$
	$\downarrow$	$\downarrow$	$\downarrow$
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σ_i :	$1 \mapsto 1$ $2 \mapsto 2$ $3 \mapsto 32$	$1 \mapsto 1$ $2 \mapsto 3$ $3 \mapsto 23$	$1 \mapsto 2$ $2 \mapsto 3$ $3 \mapsto 13$
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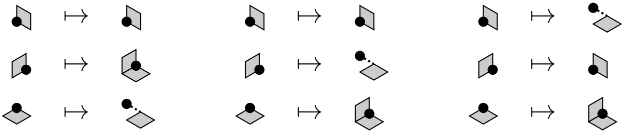
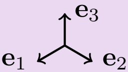
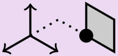
$\mathbf{E}_1^*(\sigma_i)$:



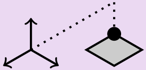
Discrete planes

Unit face $[\mathbf{x}, i]^*$, of type $i \in \{1, 2, 3\}$ at $\mathbf{x} \in \mathbb{Z}^3$





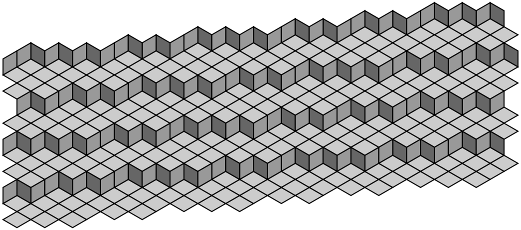
$[(-1, 1, 0), 1]^*$



$[(-3, 0, -1), 3]^*$

Discrete plane $\Gamma_{\mathbf{v}} = \{[\mathbf{x}, i]^* : 0 \leqslant \langle \mathbf{x}, \mathbf{v} \rangle < \langle \mathbf{e}_i, \mathbf{v} \rangle\}$.

$\Gamma_{(1,\sqrt{2},\sqrt{17})}$:



Dual substitutions [Arnoux-Ito 2001]

$$\sigma \quad \overset{\text{duality}}{\longmapsto} \quad \mathbf{E}_1^*(\sigma)$$

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$$\sigma \quad \overset{\text{duality}}{\longmapsto} \quad \mathbf{E}_1^*(\sigma)$$

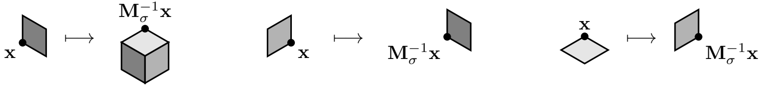
$$\mathbf{E}_1^*(\sigma)([\mathbf{x}, i]^\star) \;=\; \bigcup_{(p,j,s)\in\mathcal{A}^\star\times\mathcal{A}\times\mathcal{A}^\star \;:\; \sigma(j)=pis} [\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{P}(s)), j]^\star$$

Dual substitutions [Arnoux-Ito 2001]

$$\sigma \quad \stackrel{\text{duality}}{\longmapsto} \quad \mathbf{E}_1^*(\sigma)$$

Example: $\mathbf{E}_1^*(\sigma)$ for $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$

$$\begin{array}{lcl} [\mathbf{x}, 1]^* & \mapsto & \mathbf{M}_\sigma^{-1}\mathbf{x} + [(1,0,-1),1]^* \cup [(0,1,-1),2]^* \cup [\mathbf{0},3]^* \\ [\mathbf{x}, 2]^* & \mapsto & \mathbf{M}_\sigma^{-1}\mathbf{x} + [\mathbf{0},1]^* \\ [\mathbf{x}, 3]^* & \mapsto & \mathbf{M}_\sigma^{-1}\mathbf{x} + [\mathbf{0},2]^* \end{array}$$



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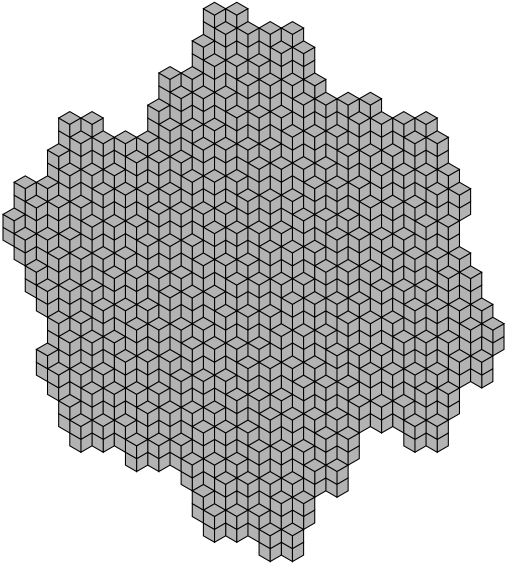
Iterating $\mathbf{E}_1^*(\sigma)$...



$\mathbf{E}_1^*(\sigma)$ + discrete planes = ♡

Proposition [Arnoux-Ito, Fernique]

$\mathbf{E}_1^*(\sigma)(\mathbf{\Gamma_v}) = \mathbf{\Gamma}_{{}^t\mathbf{M}_\sigma \mathbf{v}}$

Corollary

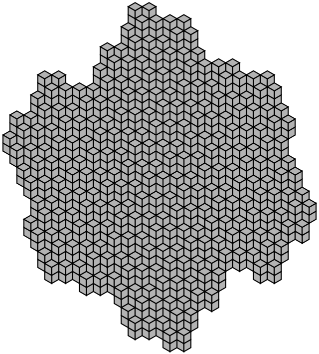
The patches $\mathbf{E}_1^*(\sigma)^n($$)$ grow within discrete planes

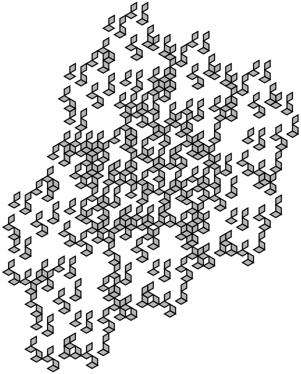
Main question

How do the $\mathbf{E}_1^*(\sigma)^n($$)$ patches grow?

1. Do they cover arbitrarily large balls?
2. Do they cover arbitrarily large balls centered at $\mathbf{0}$?

1. Do they cover **arbitrarily large balls?**

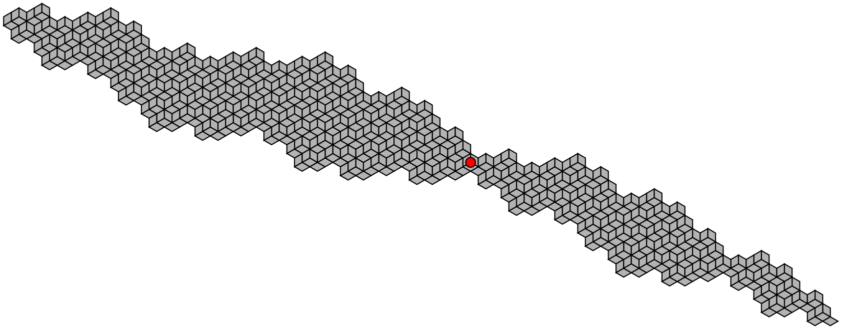




- Not always obvious...
- Links with Pisot conjecture (see later)

2. Do they cover arbitrarily large balls **centered at $\mathbf{0}$?**

Not always:



- Links with fractal topology, zero inner point, (see later)
- Links with number theory, finiteness properties (see later)

Back to Brun substitutions $(i_n)_{n\in\mathbb{N}} = 333333\ldots$

Iterating $\mathbf{E}_1^*(\sigma_3)\cdots\mathbf{E}_1^*(\sigma_3)($$)$

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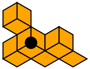

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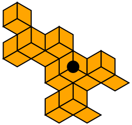

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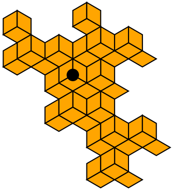

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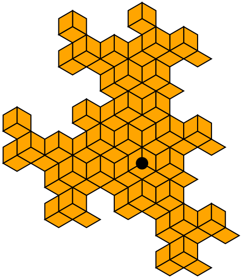

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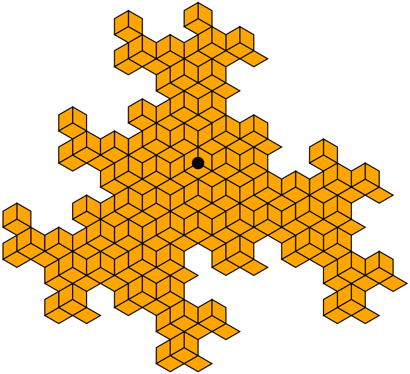
Iterating $\mathbf{E}_1^*(\sigma_3)\cdots\mathbf{E}_1^*(\sigma_3)($$)$



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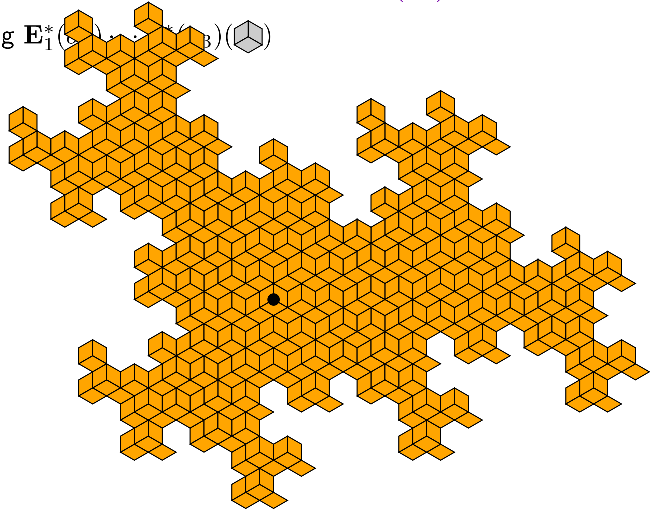

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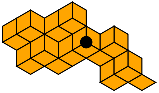

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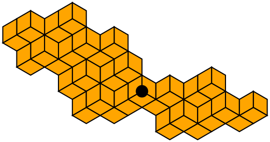

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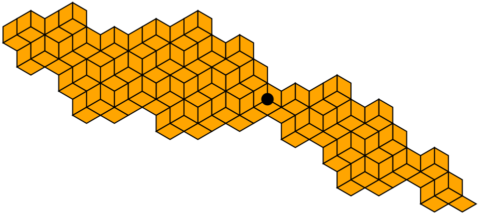

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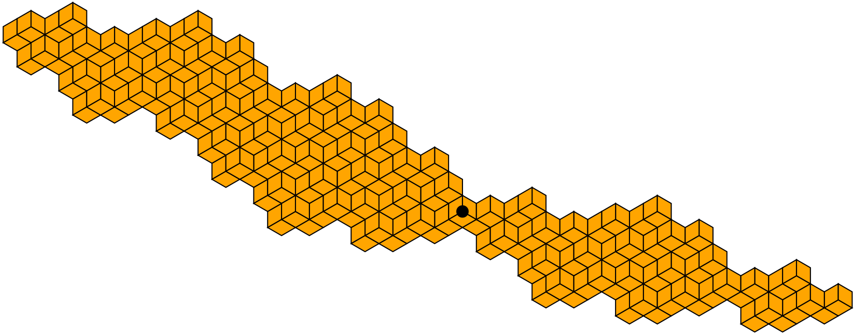

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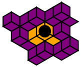

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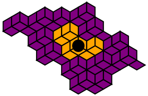

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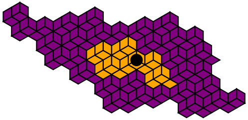

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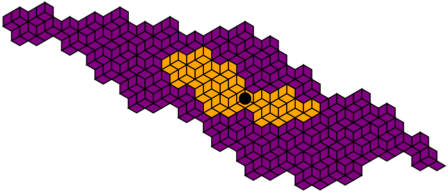

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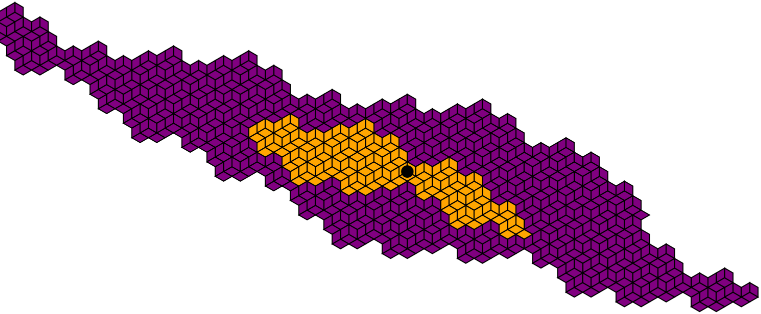

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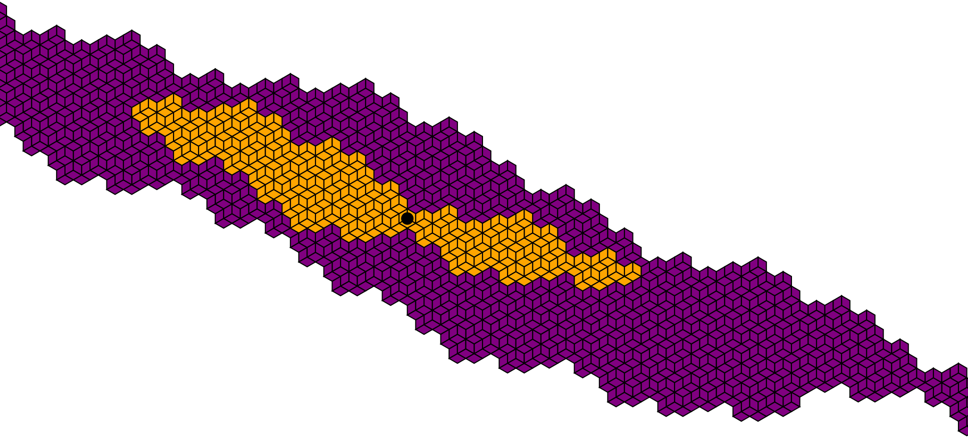

The situtuation for Brun substitutions

Let $(i_n)_{n\in\mathbb{N}} \in \{1,2,3\}^{\mathbb{N}}$ with infinitely many 3's.

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Guess

There exists a finite seed $\mathcal{V}$ such that $\mathbf{E}_1^*(\sigma_{i_1})\mathbf{E}_1^*(\sigma_{i_2})\cdots(\mathcal{V})$ contains large balls centered at $\mathbf{0}$.

Main result

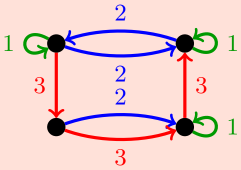
Let $(i_n)_{n\in\mathbb{N}} \in \{1,2,3\}^{\mathbb{N}}$ with infinitely many 3's.

Theorem [Berthé-Bourdon-J-Siegel]

The patterns $\mathbf{E}_1^*(\sigma_{i_1})\mathbf{E}_1^*(\sigma_{i_2})\cdots($  $)$

1. always contain arbitrarily large balls
2. contain large balls centered at $\mathbf{0}$

   $\iff$ there is an infinite path $\cdots \xrightarrow{i_2} \bullet \xrightarrow{i_1} \bullet$ in:



3. always contain large balls centered at $\mathbf{0}$
   when starting from a finite seed $\mathcal{V}$ (does not depend on (i_i))

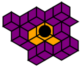
Tools

Initial idea of Ito-Ohtsuki 1994, and:

1. Annulus property
2. “Local rules”, covering properties
3. Generation graphs

Annulus property

Annulus property



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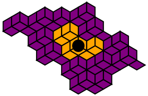

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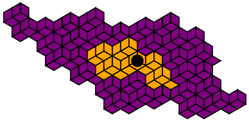

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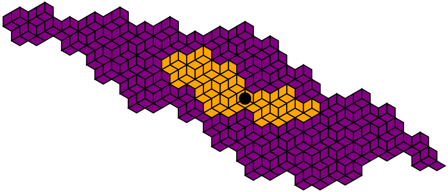

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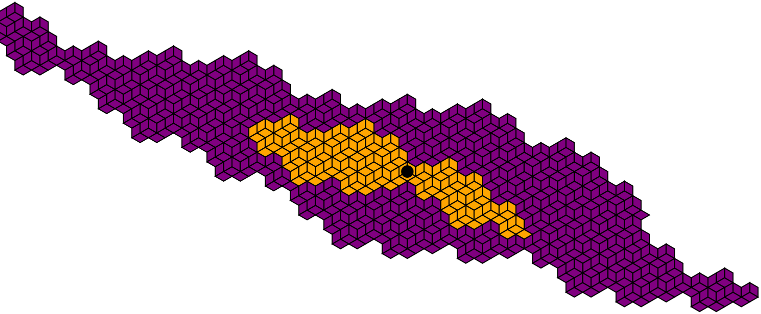

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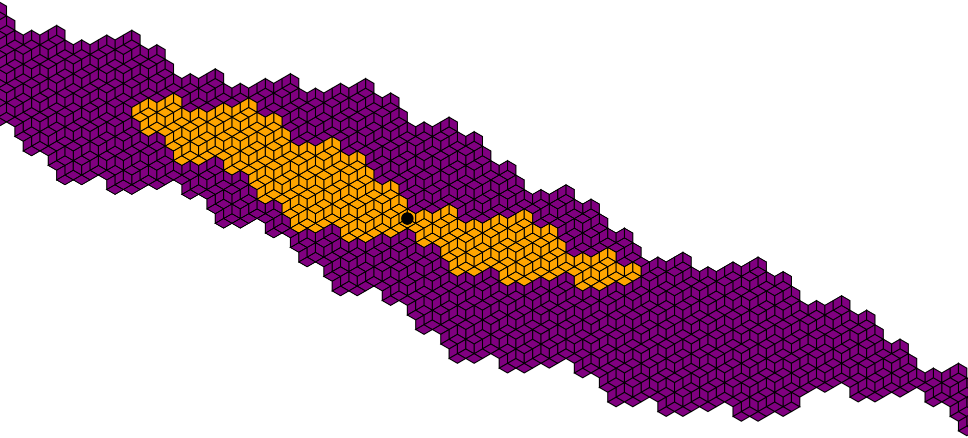
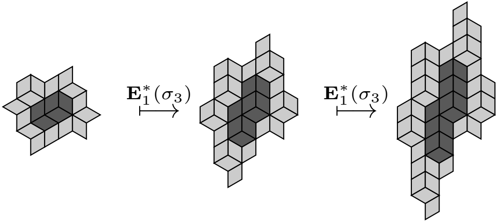


Covering properties for the annulus property

Unfortunately the annulus property **doesn't always hold**:



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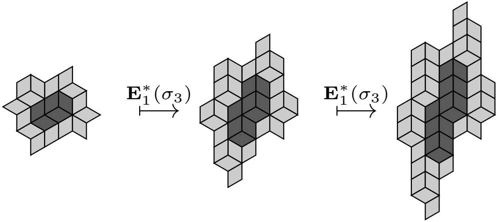

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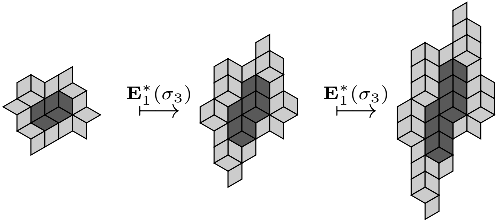

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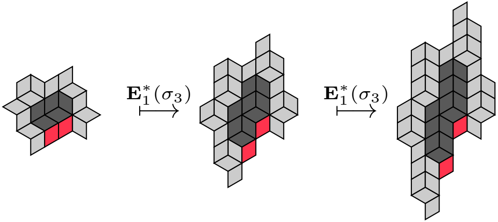

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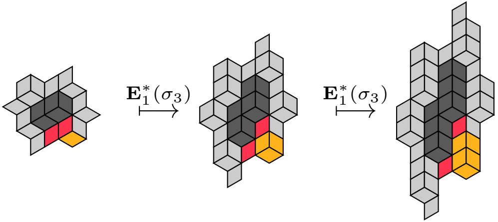

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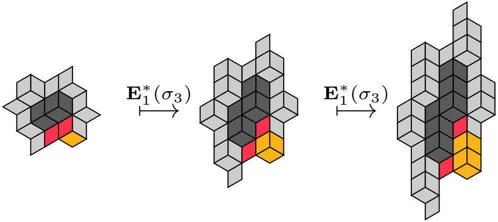

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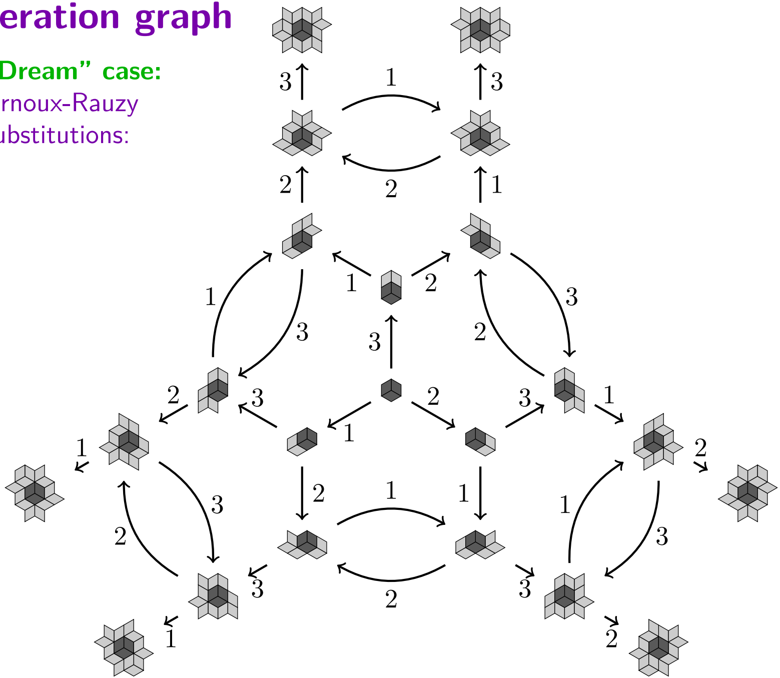
Proposition: the annulus property holds with these restrictions.

Generation graph

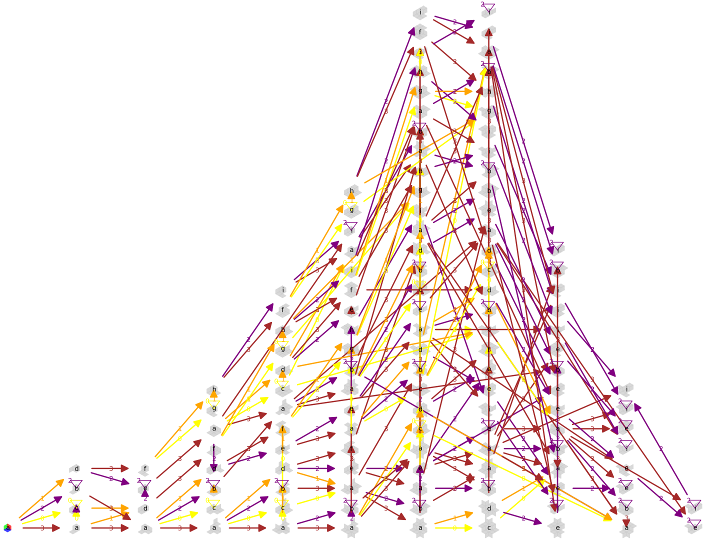
"Dream" case:
Arnoux-Rauzy substitutions:



Generation graph

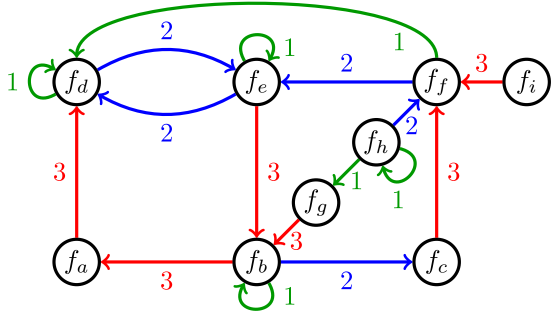
Bad approach:

Brun, Jacobi-Perron substitutions:



Generation graph

New approach:



$$
\begin{array}{lll}
f_a = [(1,1,-1),1]^\star & f_d = [(-1,1,0),2]^\star & f_g = [(-1,0,1),2]^\star \\
f_b = [(1,-1,1),3]^\star & f_e = [(-1,0,1),3]^\star & f_h = [(-1,-1,1),3]^\star \\
f_c = [(1,1,-1),2]^\star & f_f = [(-1,1,0),3]^\star & f_i = [(1,1,-1),3]^\star.
\end{array}
$$

- Full understanding of the bad language
- Allows to easily compute the finite seed

Applications: Dynamics

- **Pisot conjecture** $\Leftrightarrow$ **the** $\mathbf{E}_1^*(\sigma)^n($$)$ **contain large balls**
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- **Pisot conjecture $\Leftrightarrow$ the $\mathbf{E}_1^*(\sigma)^n$() contain large balls** [Ito-Rao 2006]
- **Hence:**

Pisot conjecture holds for products of substitutions of Brun, Arnoux-Rauzy, Jacobi-Perron, ...

Applications: topology of Rauzy fractals

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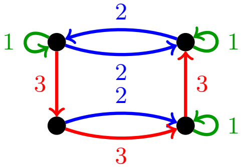

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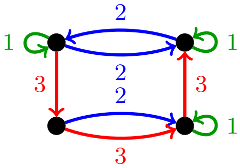

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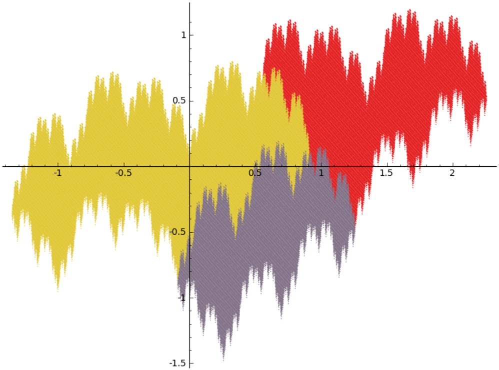

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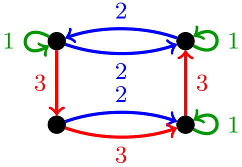

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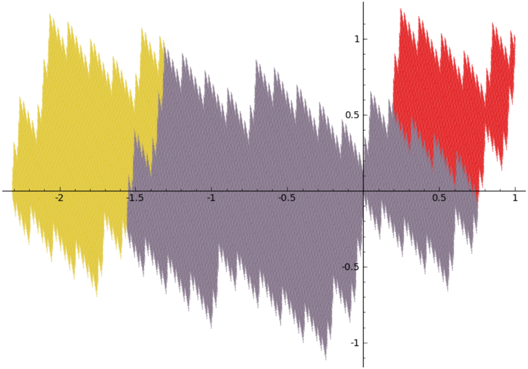

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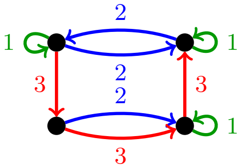

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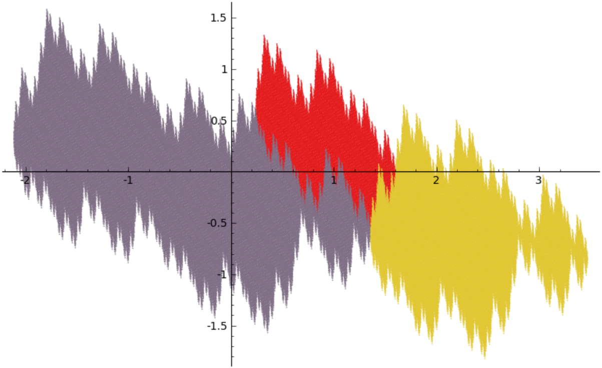

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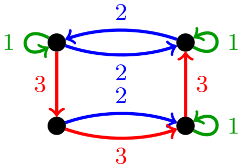

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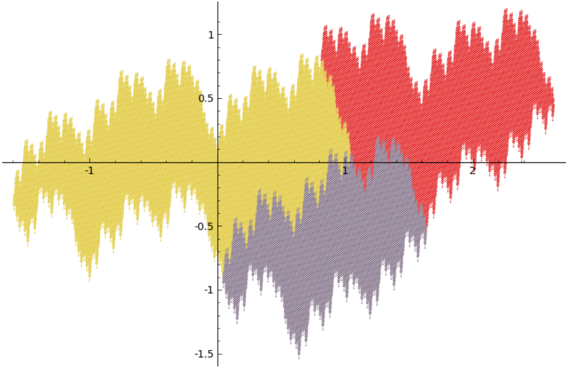

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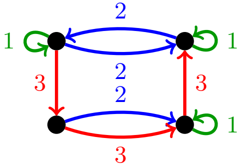

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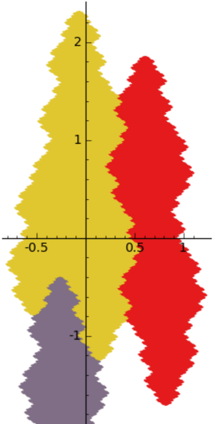

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Other topological properties:

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- **Interesting question:** which products yield simply connected fractals?

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Question

Computer experiments suggest:
The Pisot eigenvalue of $\mathbf{M}_{i_1} \cdots \mathbf{M}_{i_n}$ is **totally real**
when $i_1 \cdots i_n$ is in the language.

Why?

Conclusion and perspectives

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Thank you for your attention