Self-affine sets with nonempty interior

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Joint work with **Jarkko Kari**

Affine iterated function systems

Let $f_1, \ldots, f_n : \mathbb{R}^d \to \mathbb{R}^d$ be contracting maps

Theorem (Hutchinson 1981)

There is a unique nonemtpy compact set $X \subseteq \mathbb{R}^d$ such that

$$X = f_1(X) \cup \cdots \cup f_n(X).$$

- $f_1, \ldots, f_n$ is an **iterated function system (IFS)**
- We will restrict to **affine** maps $f_i : x \mapsto A_i x + v_i$

Self-affine tiles

- Let $A \in \mathcal{M}_d(\mathbb{Z})$ be an expanding matrix (eigenvalues $|\lambda_i| > 1$)
- Let $\mathcal{D} \subseteq \mathbb{Z}^d$

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- **Question:** When does X have nonemtpy interior?

Example 1

- $A = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$
- $\mathcal{D} = \left\{ \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \right\}$

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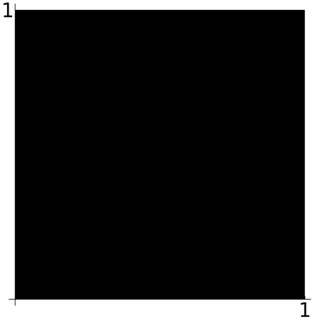

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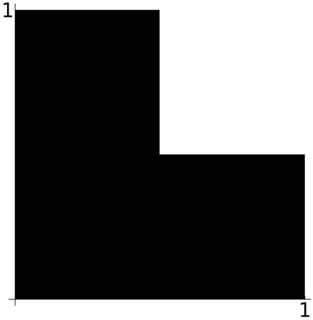

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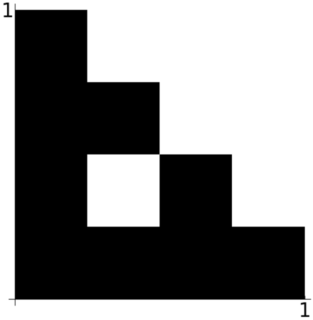

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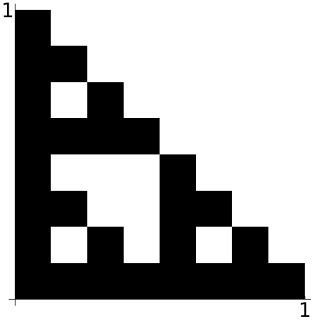

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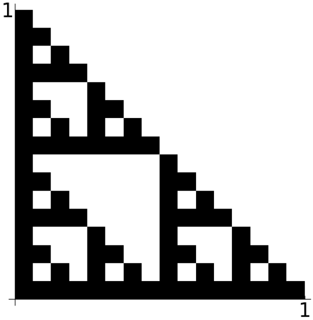

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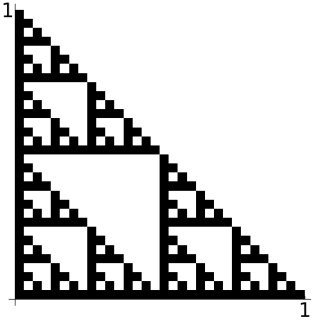

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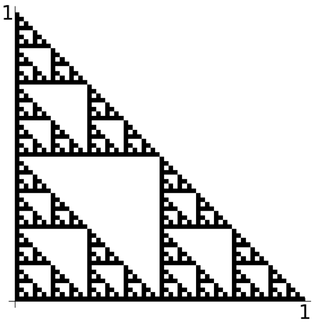

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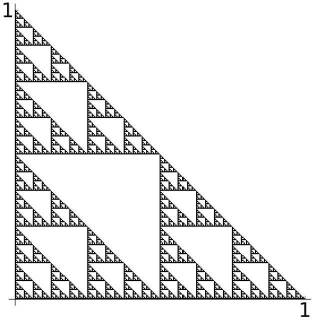

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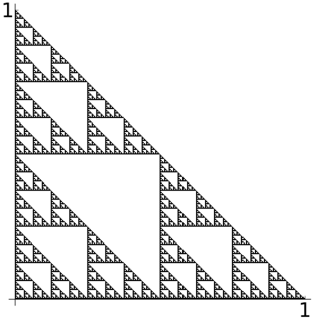

We have $4\mu(X) = \mu(X) + \mu(X) + \mu(X) = 3\mu(X)$,
so $\mu(X) = 0$ and X has **empty interior**

Necessary condition

- $A \in \mathcal{M}_d(\mathbb{Z})$ is expanding
- $\mathcal{D} \subseteq \mathbb{Z}^d$

We will now assume that $|\mathcal{D}| = \det(A)$

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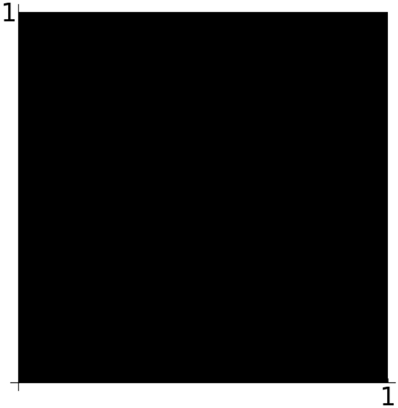

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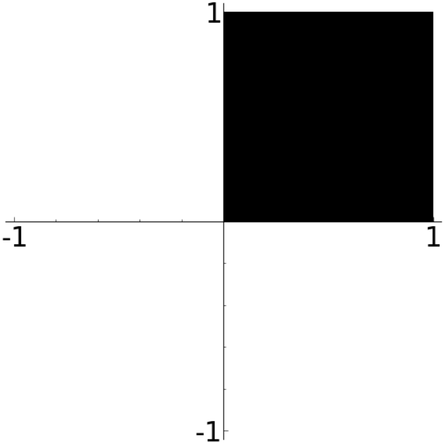

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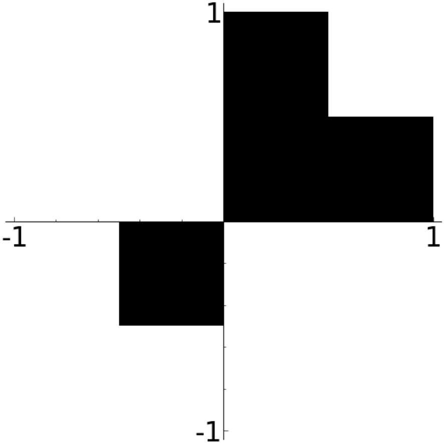

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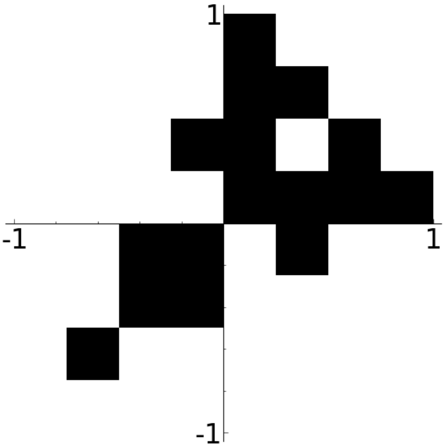

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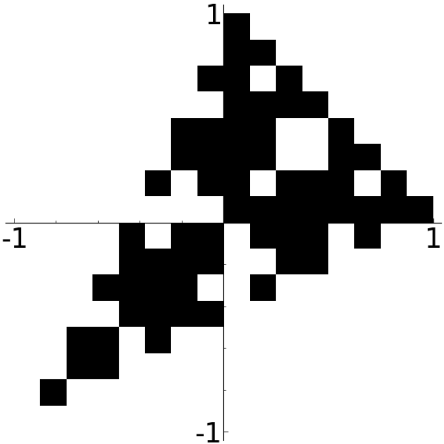

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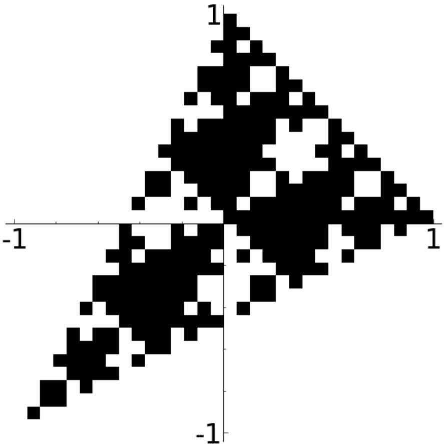

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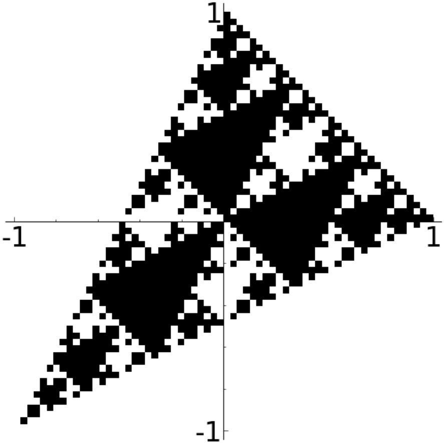

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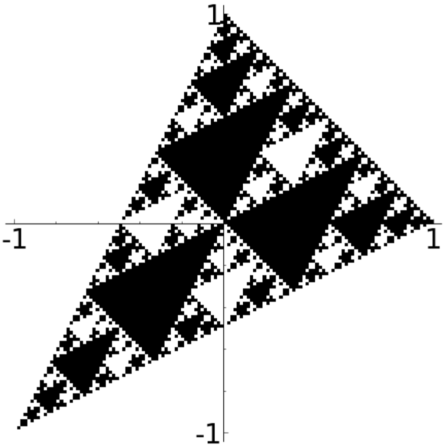

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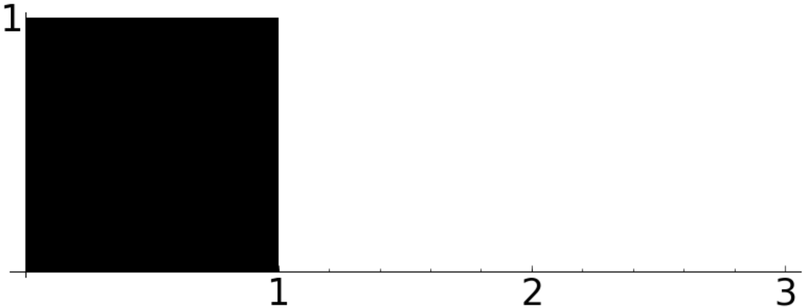

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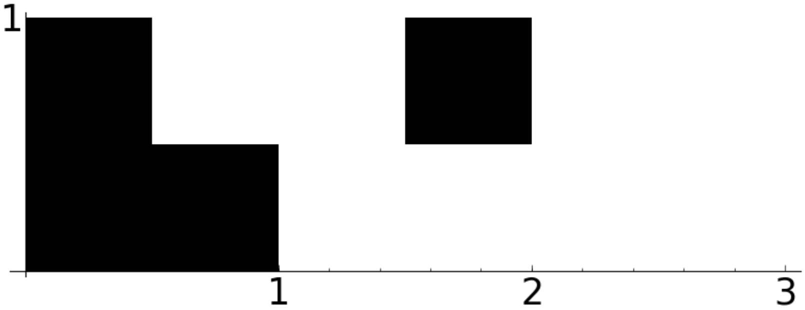

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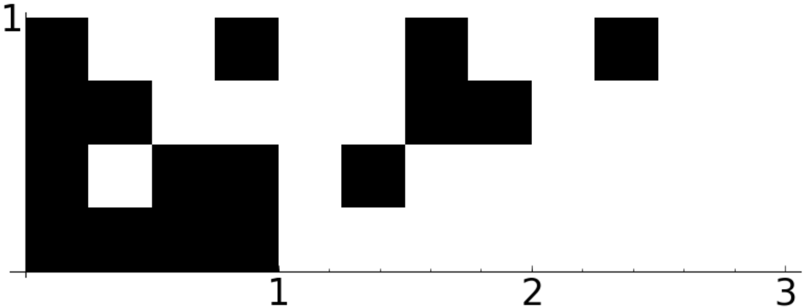

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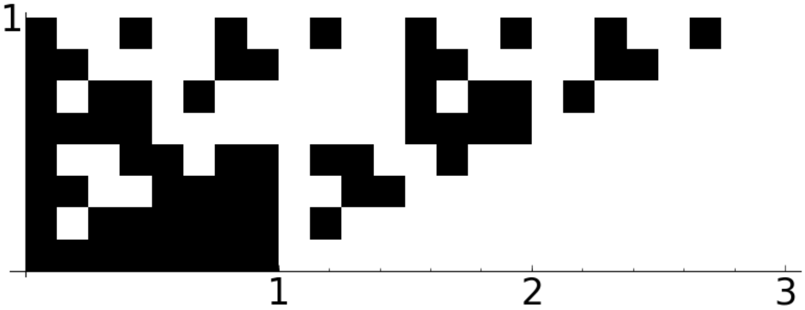

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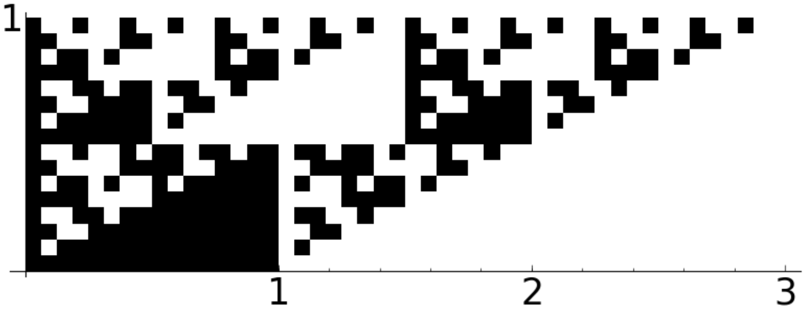

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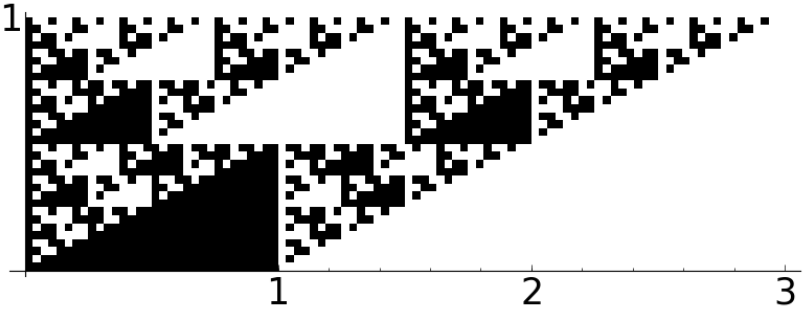

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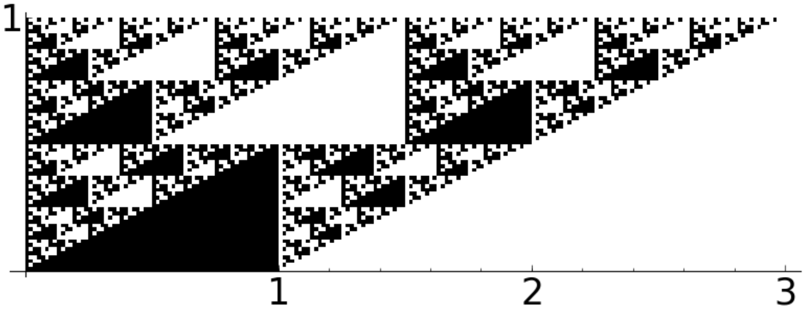

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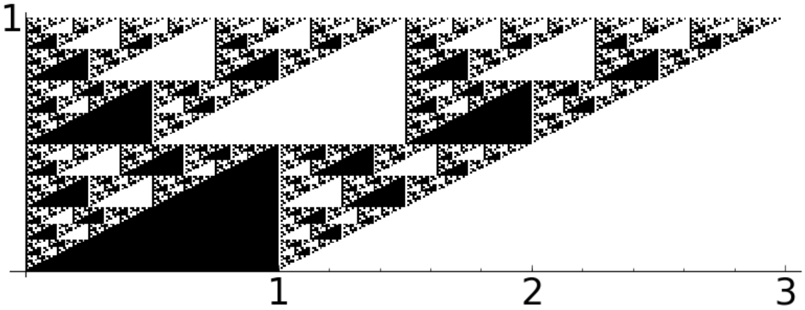

A sufficient condition

Theorem (Bandt 1991)

$\mathcal{D}$ is a **complete set of representatives in** $\mathbb{Z}^d/A\mathbb{Z}^d$

$\implies X$ has nonempty interior

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Previous "Sierpiński" examples:

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What happens when $\mathcal{D}$ is not complete?

Examples with "incomplete" $\mathcal{D}$

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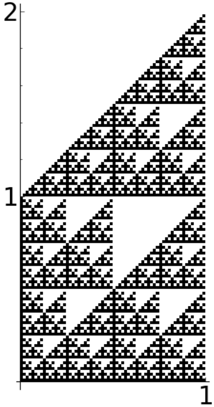

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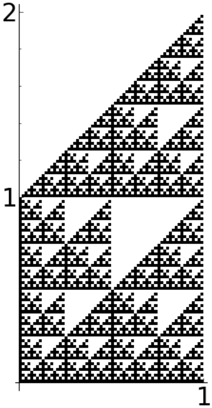

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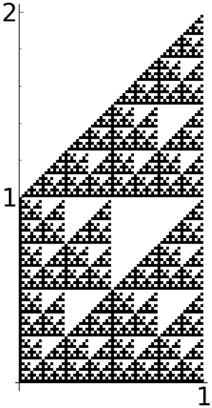

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(why?)

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so

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} X + \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$$

Examples with "incomplete" $\mathcal{D}$

Indeed,

$$\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) X = X + \left\{ \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) \right\},$$

so

$$\begin{aligned} \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right) X &= \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) X + \left\{ \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right) \right\} \\ &= X + \left\{ \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) \right\} \\ &\qquad + \left\{ \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right) \right\} \end{aligned}$$

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Examples with “incomplete” $\mathcal{D}$

Indeed,

$$\begin{pmatrix}2&0\\0&2\end{pmatrix}X = X + \left\{\binom{0}{0}, \binom{1}{0}, \binom{0}{1}, \binom{1}{2}\right\},$$

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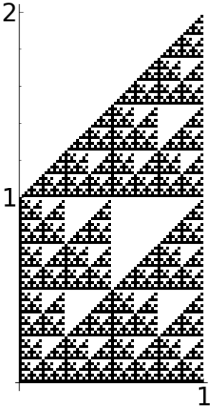
so $16\mu(X) \leqslant 15\mu(X)$.

Examples with "incomplete" $\mathcal{D}$

$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
$\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$



X **has empty interior**

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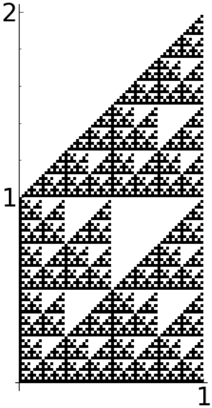

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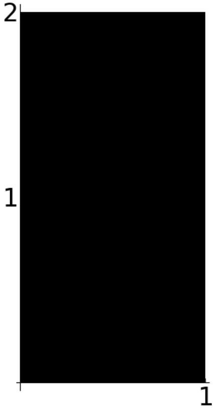

X **has nonempty interior**

Summary

Original question: when does X have nonempty interior?

- Self-affine tiles
  - $\mathcal{D}$ is complete: sufficient condition
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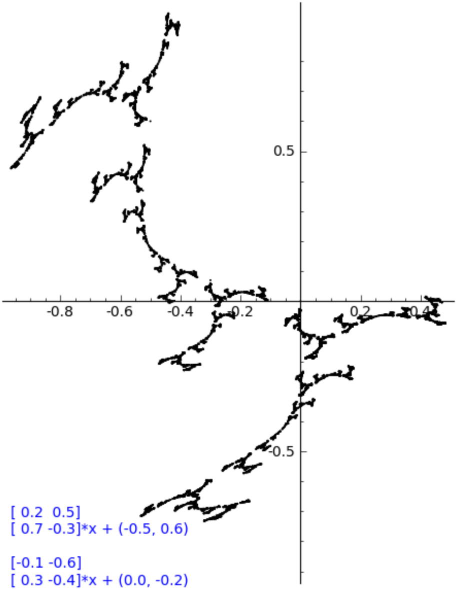
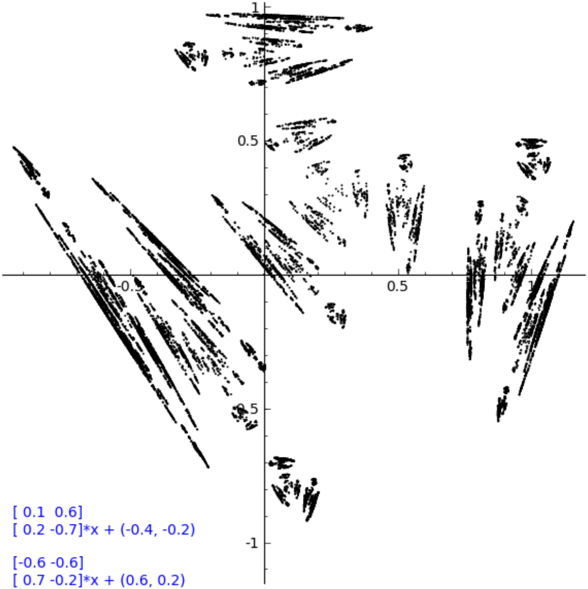
  - $\mathcal{D}$ is not complete: anything can happen
  - There is an **algorithm**: [Gabardo-Yu 2006] and [Bondarenko-Kravchenko 2011]
- More general families? (When matrices A_i are not equal?)

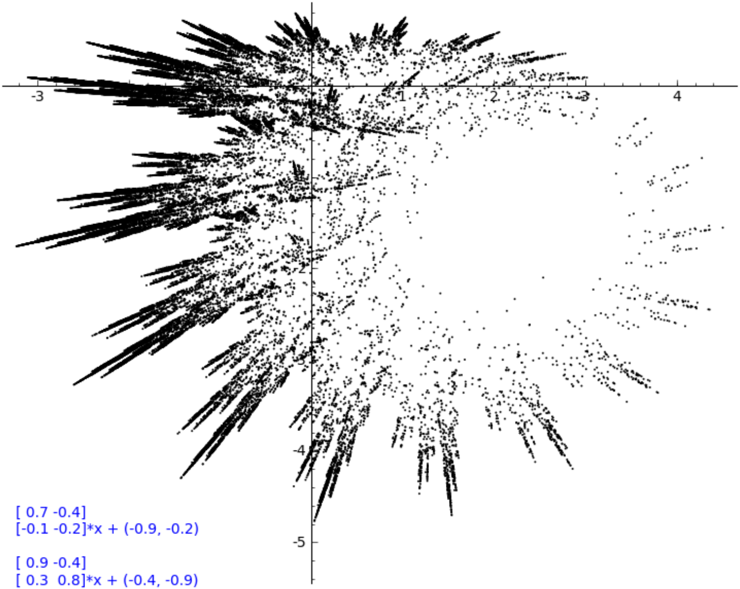
Affine iterated function systems



Affine iterated function systems



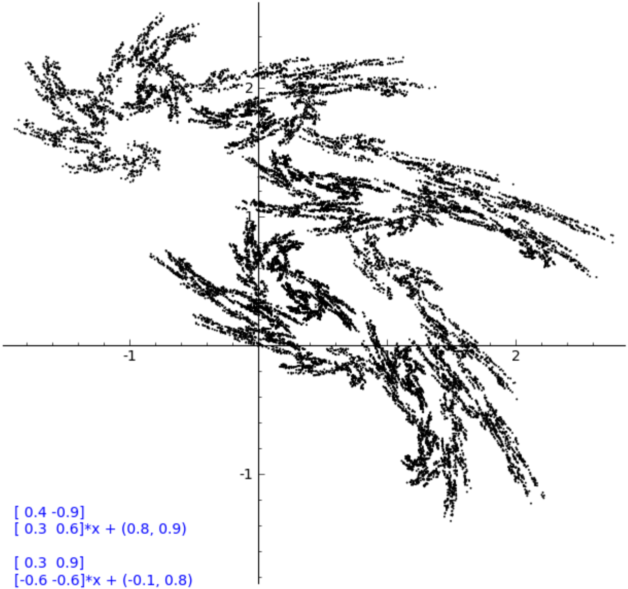
Affine iterated function systems



[0.7 -0.4]
[-0.1 -0.2]*x + (-0.9, -0.2)

[0.9 -0.4]
[0.3 0.8]*x + (-0.4, -0.9)

Affine iterated function systems



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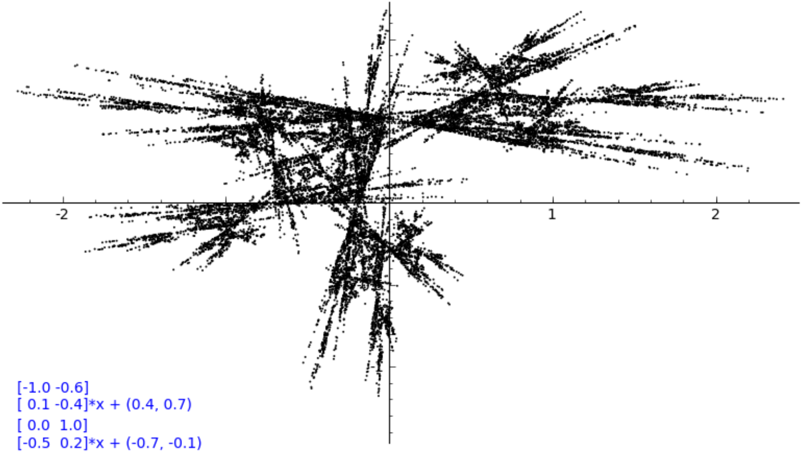
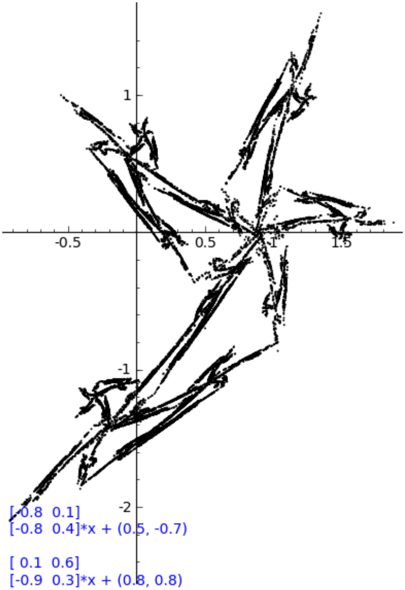


[-1.0 -0.6]
[0.1 -0.4]*x + (0.4, 0.7)

[0.0 1.0]
[-0.5 0.2]*x + (-0.7, -0.1)

Affine iterated function systems



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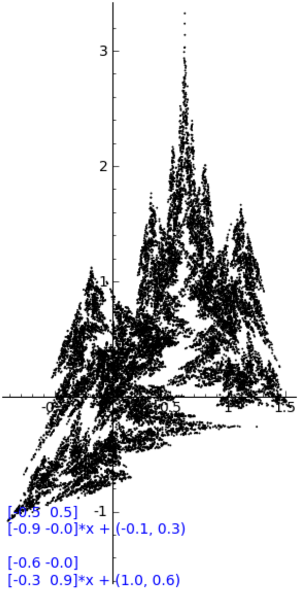

Graph-directed IFS (GIFS)

More general: replace $X = f_1(X) \cup \cdots \cup f_k(X)$ by a **system of several unknowns** $X_1, \ldots, X_n$.

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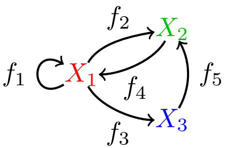
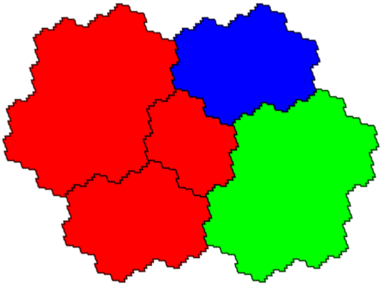
$$\begin{cases} X_1 = f_1(X_1) \cup f_2(X_2) \cup f_3(X_3) \\ X_2 = f_4(X_1) \\ X_3 = f_5(X_2) \end{cases}$$





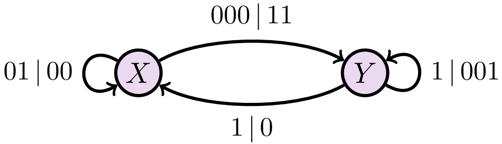
GRAPH IFS
with 3 states

Computational tools: multitape automata

Multitape automata

d-tape automaton:

- alphabet $\mathcal{A} = A_1 \times \cdots \times A_d$
- states $\mathcal{Q}$
- transitions $\mathcal{Q} \times (A_1^+ \times \cdots \times A_d^+) \to \mathcal{Q}$



$\mathcal{A} = \{0,1\} \times \{0,1\}$
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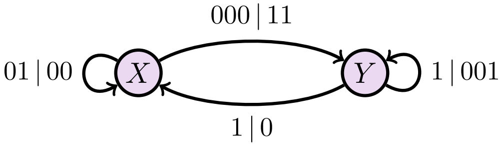

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Accepted configurations:

000
11

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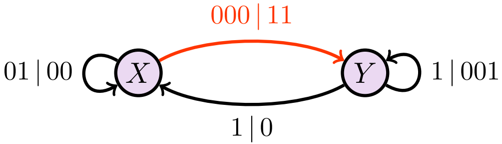

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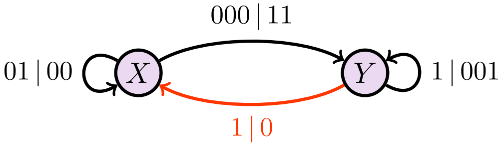

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0001
110

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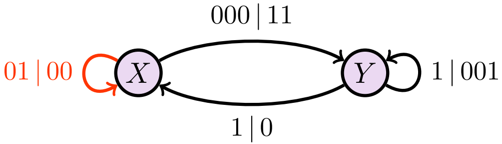

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11000

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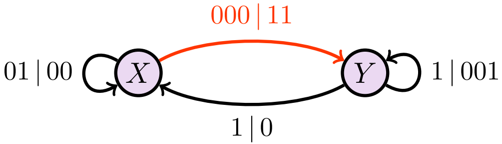

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000101000
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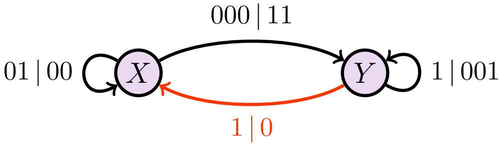

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0001010001
11000110

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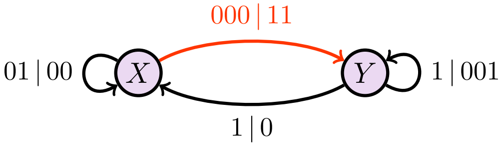

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0001010001000
1100011011

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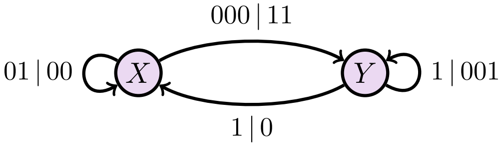

$\mathcal{A} = \{0,1\} \times \{0,1\}$
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Accepted configurations:

$$\begin{matrix} 0001010001000\ldots \\ 1100011011\ldots \end{matrix} \in \mathcal{A}^{\mathbb{N}} = (\{0,1\} \times \{0,1\})^{\mathbb{N}}$$

2-tape automaton $\longmapsto$ 2-dimensional IFS

Transition 

Tape alphabets A_1, A_2

2-tape automaton $\longmapsto$ 2-dimensional IFS

Transition 

Tape alphabets A_1, A_2

$\longmapsto$ Mapping $f\binom{x}{y} = \begin{pmatrix} |A_1|^{-|u|} & 0 \\ 0 & |A_2|^{-|v|} \end{pmatrix} \binom{x}{y} + \begin{pmatrix} 0.u_1 \dots u_{|u|} \\ 0.v_1 \dots v_{|v|} \end{pmatrix}$

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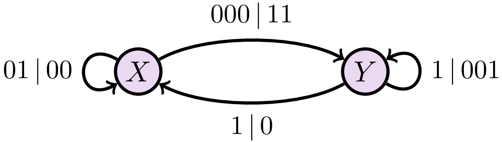
Automaton:



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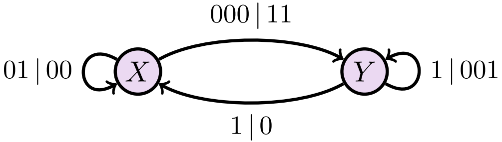
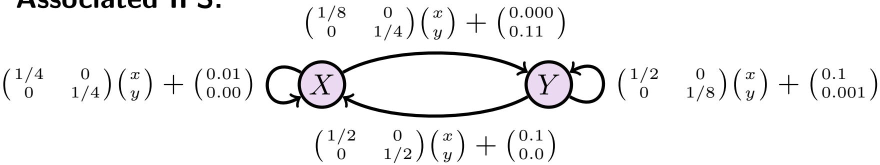
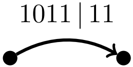


Associated IFS:

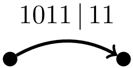


2-tape automaton $\longmapsto$ 2-dimensional IFS

- 

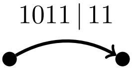
- $f\binom{x}{y} \quad = \quad \begin{pmatrix} 1/16 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0.x_1 x_2 \ldots \\ 0.y_1 y_2 \ldots \end{pmatrix} + \begin{pmatrix} 0.1011 \\ 0.11 \end{pmatrix}$

2-tape automaton $\longmapsto$ 2-dimensional IFS

- 

- $$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1/16 & 0\\0 & 1/4\end{pmatrix}\begin{pmatrix}0.x_1x_2\dots\\0.y_1y_2\dots\end{pmatrix} + \begin{pmatrix}0.1011\\0.11\end{pmatrix}$$
$$= \begin{pmatrix}0.0000x_1x_2\dots\\0.00y_1y_2\dots\end{pmatrix} + \begin{pmatrix}0.1011\\0.11\end{pmatrix}$$

2-tape automaton $\longmapsto$ 2-dimensional IFS

- 

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$$= \begin{pmatrix} 0.0000x_1x_2\ldots \\ 0.00y_1y_2\ldots \end{pmatrix} + \begin{pmatrix} 0.1011 \\ 0.11 \end{pmatrix}$$
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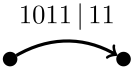
- 

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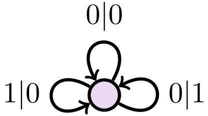
Key correspondence

Fractal associated with automaton $\mathcal{M}$

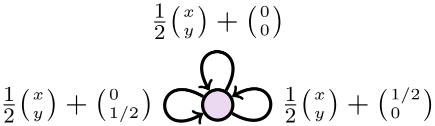
$$=$$

$$\left\{ \begin{pmatrix} 0.x_1x_2\ldots \\ 0.y_1y_2\ldots \end{pmatrix} \in \mathbb{R}^2 \ : \ \begin{pmatrix} x_1x_2\ldots \\ y_1y_2\ldots \end{pmatrix} \text{ accepted by } \mathcal{M} \right\}$$

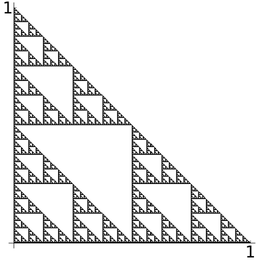
Example: Sierpiński triangle



Automaton language



Iterating IFS maps



$$= \left\{ \begin{pmatrix} 0.x_1 x_2 \dots \\ 0.y_1 y_2 \dots \end{pmatrix} : (x_n, y_n) \neq (1,1),\ \forall n \geqslant 1 \right\}$$

Multitape automata $\longleftrightarrow$ fractals

Language theoretical properties of accepted language

$$\longleftrightarrow$$

Topological properties of fractal set

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Example (in 2D) [Dube 1993, original idea]

Automaton accepts one word of the form $\binom{0.x_1x_2\ldots}{0.x_1x_2\ldots}$

$\iff X$ intersects the diagonal $\{(x,x) : x \in [0,1]\}$

Language universality $\iff$ nonempty interior

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Fact 2: $\mathcal{M}$ is prefix-universal $\iff$ X has nonempty interior

- **Universal with prefix** 1 (but not universal)
  1D, 1 state, transitions 1, 10, 00

$$f_1(x) = x/2 + 1/2$$
$$f_2(x) = x/4$$
$$f_3(x) = x/4 + 1/2$$

[redacted] [redacted] [redacted]

Main result

Theorem [J-Kari 2013]

For 3-state, 2-tape automata:

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Corollary

For 2D affine graph-IFS with 3 states:

- $= [0,1]^2$ is undecidable
- empty interior is undecidable

Proof idea

Post correspondence problem (undecidable)

Given n pairs of words $(u_1, v_1), \ldots, (u_n, v_n)$,
does there exists $i_1, \ldots, i_k$
such that $u_{i_1} \cdots u_{i_k} = v_{i_1} \cdots v_{i_k}$?

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Examples:

- There is a solution:
  $(u_1, v_1) = (aa, aab)$, $(u_2, v_2) = (bb, ba)$,
  $(u_3, v_3) = (abb, b)$

  $(i_1, i_2, i_3, i_4) = (1, 2, 1, 3)$ is a solution because

  $$u_1 u_2 u_1 u_3 = aa\ bb\ aa\ abb = aab\ ba\ aab\ b = v_1 v_2 v_1 v_3$$

- No solution exists:
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Variant: infinite-PCP

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Theorem [Dube 1993]

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  $\iff$ attractor contains a point $\binom{0.x_1 x_2 \ldots}{0.x_1 x_2 \ldots}$
  $\iff$ attractor $\cap$ diagonal $\neq \varnothing$
- **So:** "attractor $\cap$ diagonal $\neq \varnothing$" is undecidable

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- Universality: reduce PCP
- Prefix-universality: reduce a variant of PCP (“prefix-PCP”)

Conclusion & perspectives

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- Links automata $\leftrightarrow$ topology:

Property of the d-tape automaton	Topological property
$\exists$ configurations with $=$ tapes	Intersects the diagonal [Dube]
Is universal	Is equal to $[0,1]^d$
Has universal prefixes	Has nonempty interior
?	Is connected
?	Is totally disconnected
Compute language entropy	Compute fractal dimension

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Thank you

References

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