Toral automorphisms, Markov Partitions and fractals

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University of Turku, Finland

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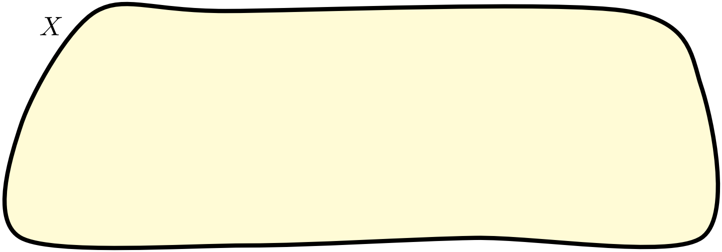
Mathematics department seminar
中山大学
广州, 中国

Aim: coding orbits of $T : X \to X$



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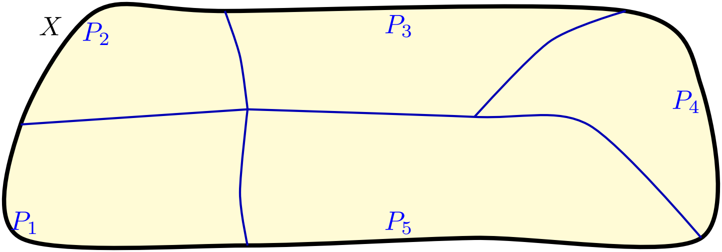

Partition $\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5\}$

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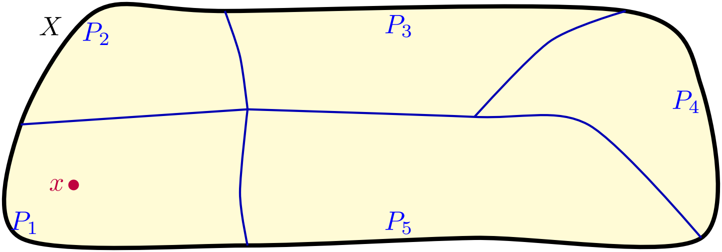

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Coding of x : $\quad 1$

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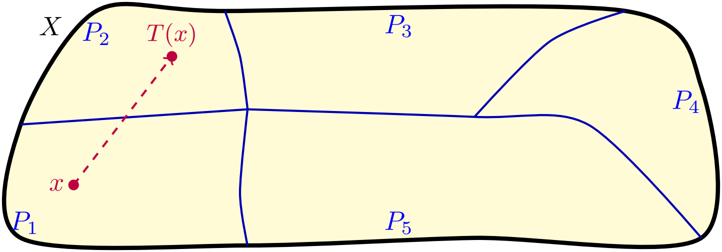

Partition $\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5\}$

Coding of x : $\quad 1 \; 2$

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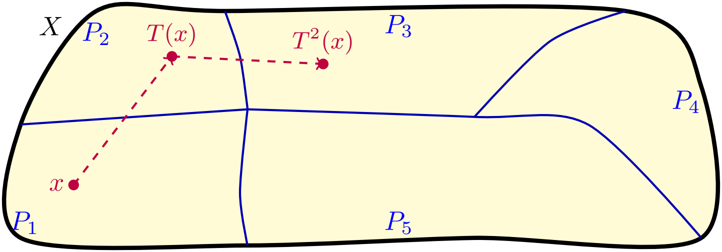

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Coding of x : $1\ 2\ 3$

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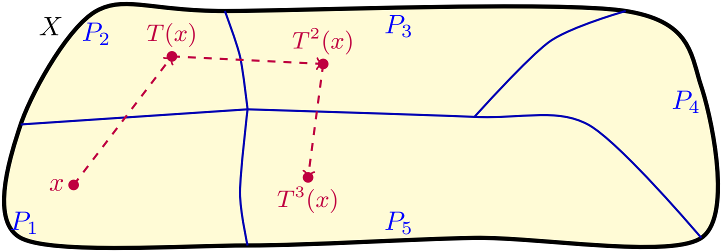

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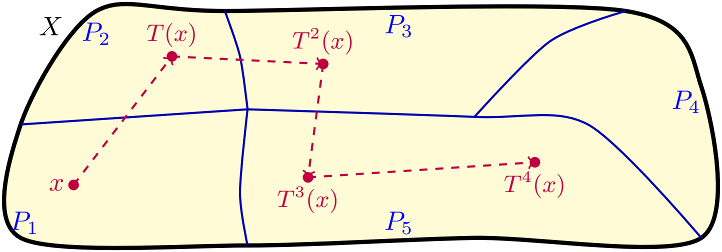

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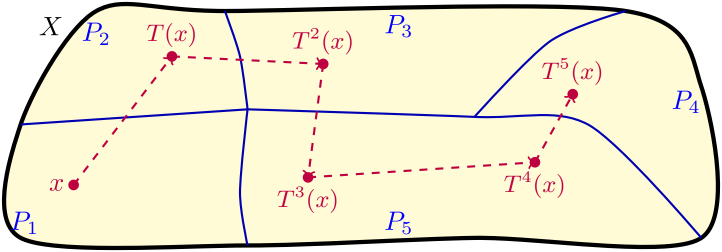

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Coding of x : $\quad 1\ 2\ 3\ 5\ 5\ 4$

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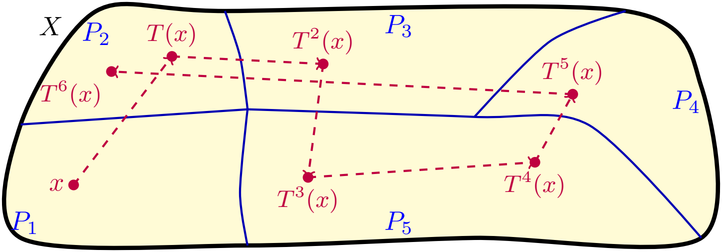

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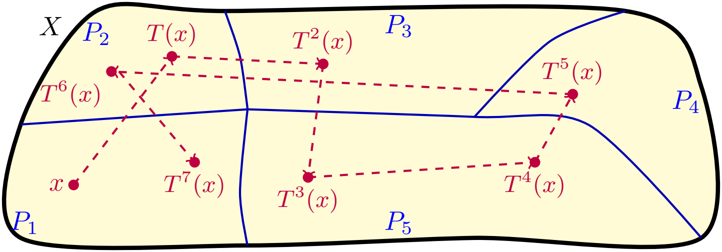

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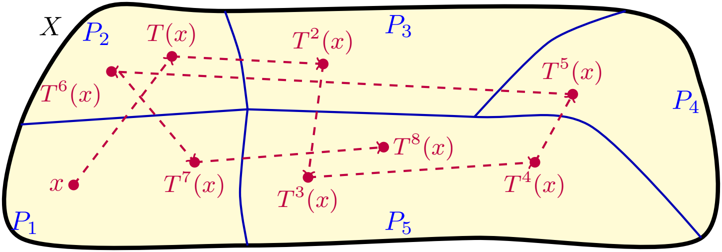

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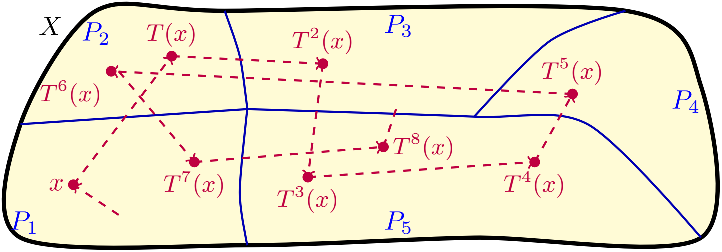

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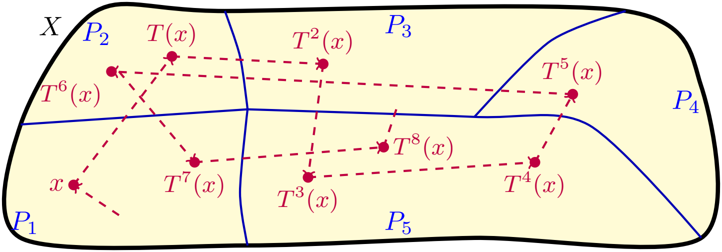

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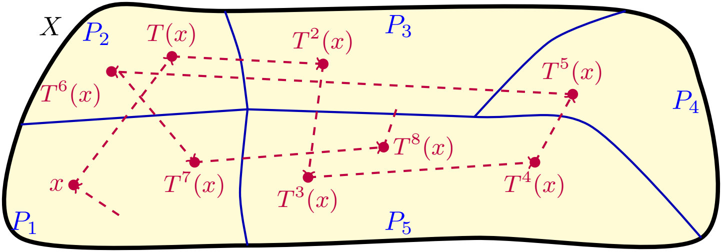

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($\Sigma_{\mathcal{P}}$, shift letter) : *symbolic dynamical system*

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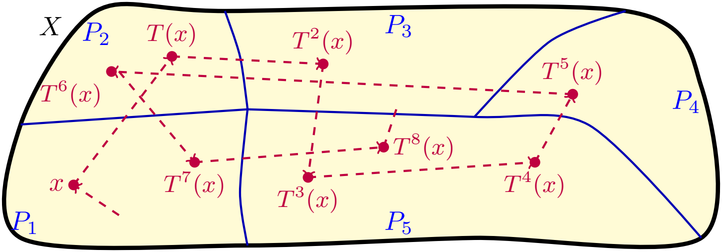

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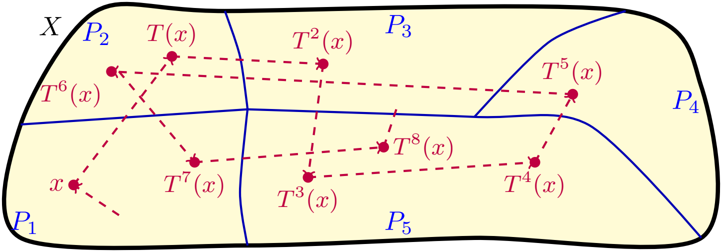

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➥ Partition $\mathcal{P}$ must be well chosen!

Coding orbits of $T : X \to X$

- $\mathcal{P}$ is **well chosen** if :
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x & \in & P_{x_0} \\
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- **Then we have:** a **symbolic representation** of (X, T) by $\mathcal{P}$:

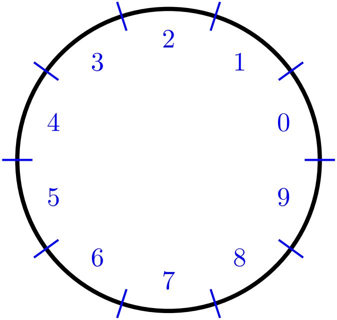
$$\begin{tikzcd} \Sigma_{\mathcal{P}} \arrow[r, "\text{décalage}"] \arrow[d, "\varphi"'] & \Sigma_{\mathcal{P}} \arrow[d, "\varphi"] \\ X \arrow[r, "T"'] & X \end{tikzcd}$$

$$\varphi : (x_n) \mapsto \text{l'unique } x \in \bigcap_{n=-\infty}^{\infty} T^{-n}(P_{x_n})$$

φ continue et surjective
(pas forcément injective)

Simplest example : multiply by 10 in $[0, 1]$

$X = [0, 1]$ $T : x \mapsto 10x \pmod 1$ $\mathcal{P} = \left\{ \left] \frac{i}{10}, \frac{i+1}{10} \right[: 0 \leqslant i \leqslant 9 \right\}$



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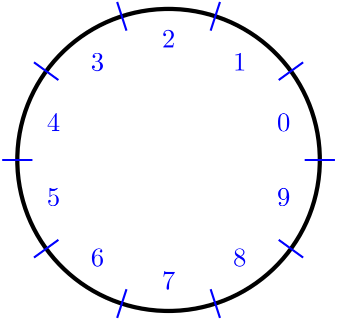

Orbit of $\pi - 3$:

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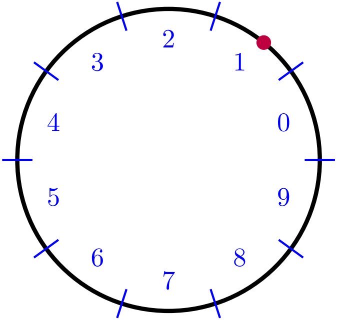

Orbit of $\pi - 3$:

0.1

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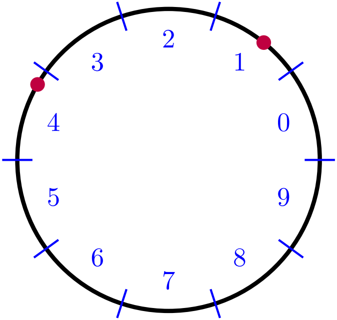

Orbit of $\pi - 3$:

0.14

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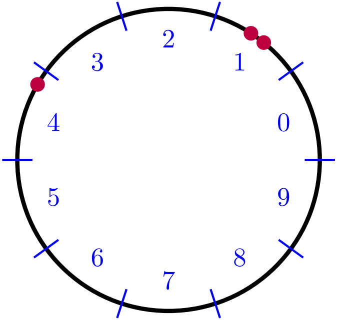

Orbit of $\pi - 3$:

0.141

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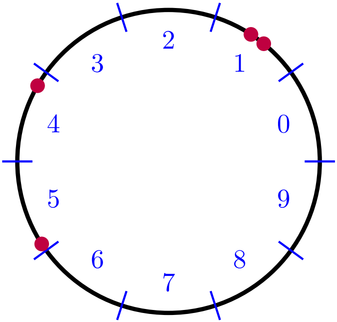

Orbit of $\pi - 3$:

0.1415

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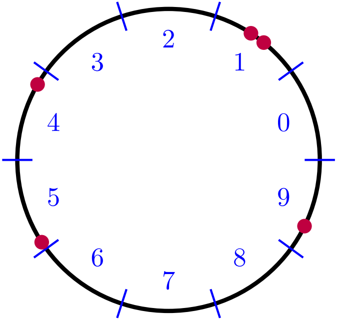

Orbit of $\pi - 3$:

0.14159

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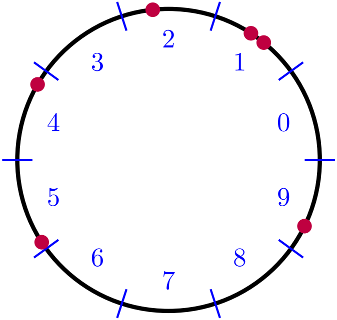

Orbit of $\pi - 3$:

0.141592

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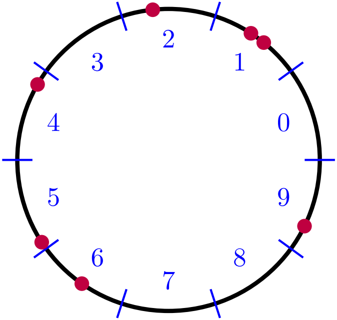

Orbit of $\pi - 3$:

0.1415926

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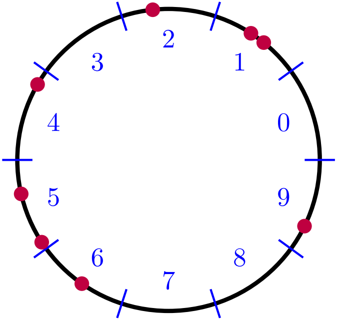

Orbit of $\pi - 3$:

0.14159265

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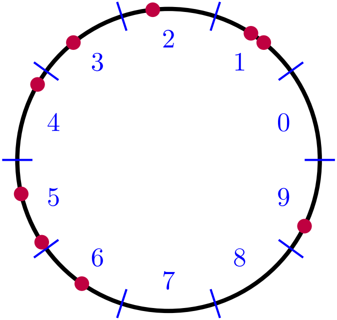

Orbit of $\pi - 3$:

0.141592653

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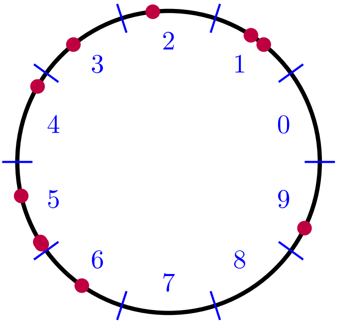

Orbit of $\pi - 3$:

0.1415926535

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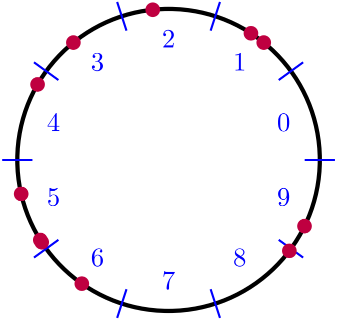

Orbit of $\pi - 3$:

0.14159265358

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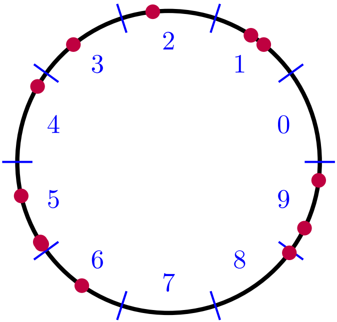

Orbit of $\pi - 3$:

0.141592653589

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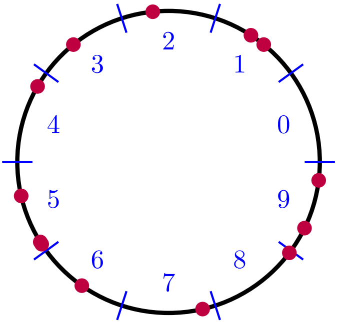

Orbit of $\pi - 3$:

0.1415926535897

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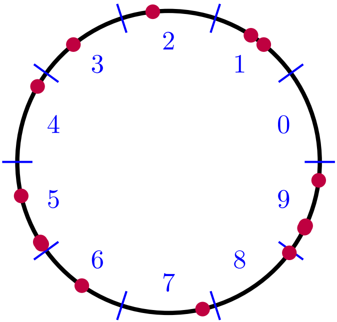

Orbit of $\pi - 3$:

$0.14159265358979312\cdots$

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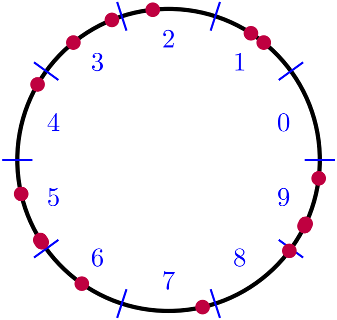

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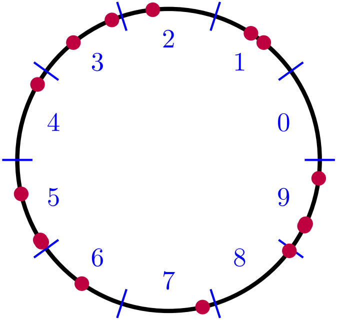
Codings $\Longleftrightarrow$ decimal expansions
Valid codings: $\Sigma_{\mathcal{P}} = \{0, \ldots, 9\}^{\mathbb{Z}}$

Simplest example : multiply by 10 in $[0,1]$

$X = [0,1] \qquad T : x \mapsto 10x \pmod 1 \qquad \mathcal{P} = \left\{ \left] \frac{i}{10}, \frac{i+1}{10} \right[: 0 \leqslant i \leqslant 9 \right\}$

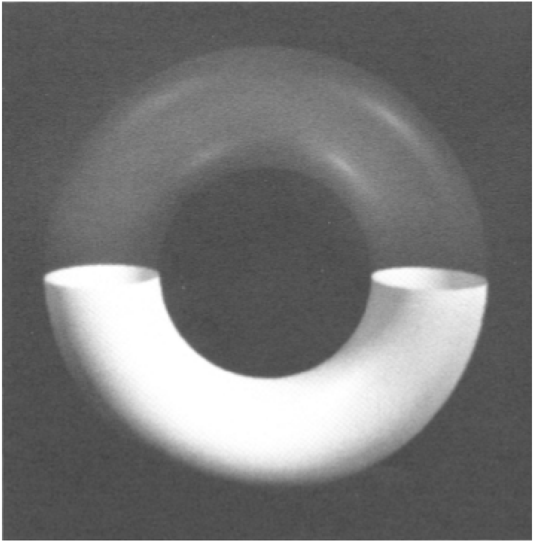


Orbit of $\pi - 3$:
$0.14159265358979312\cdots$

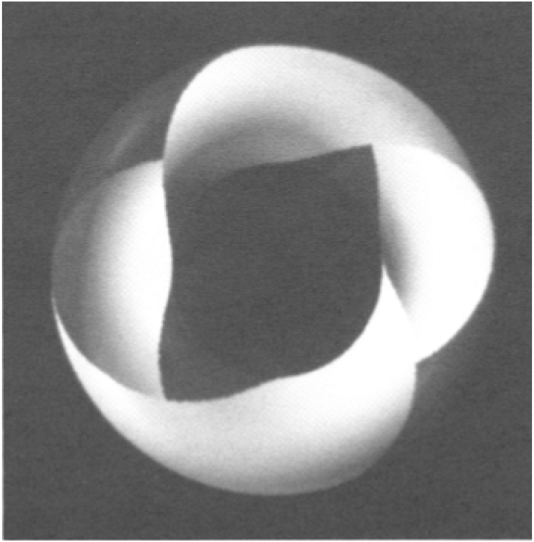
Codings $\Longleftrightarrow$ decimal expansions
Valid codings: $\Sigma_{\mathcal{P}} = \{0, \ldots, 9\}^{\mathbb{Z}}$

Remark: coding $\varphi : \Sigma_{\mathcal{P}} \to X$ is not injective:
$0.999\cdots = 1.000\cdots$ or $0.46999\cdots = 0.47000\cdots$
(every decimal number has two preimages)

Toral automorphisms

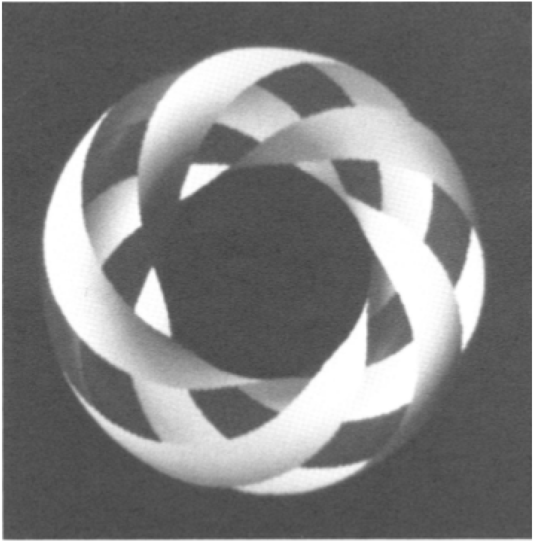


Images : *Visualizing Toral Automorphisms*, M. Grayson, B. Kitchens, G. Zettler



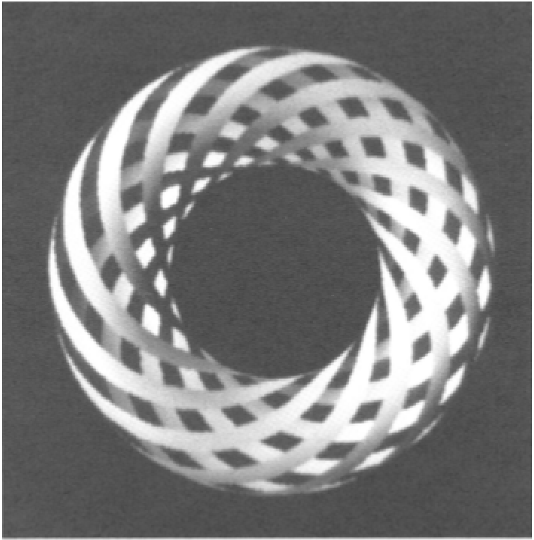
$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^1$$

Images : *Visualizing Toral Automorphisms*, M. Grayson, B. Kitchens, G. Zettler



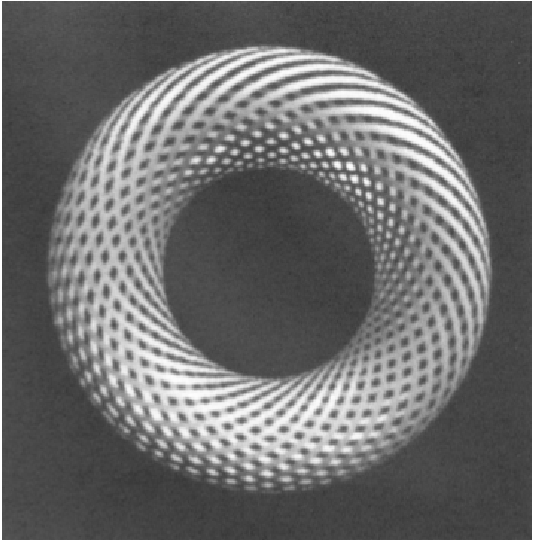
$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^2$$

Images : *Visualizing Toral Automorphisms*, M. Grayson, B. Kitchens, G. Zettler



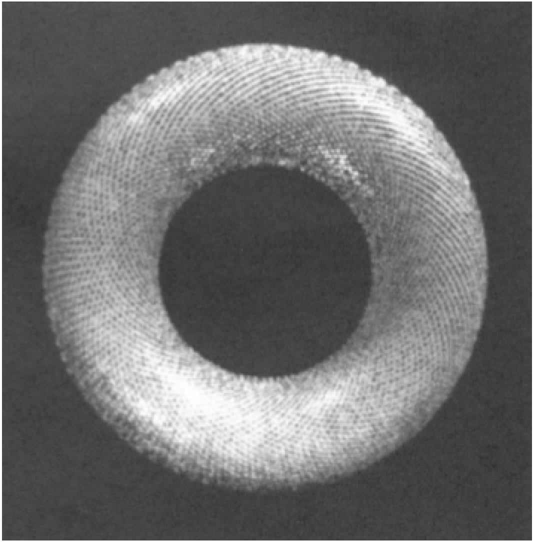
$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^3$$

Images : *Visualizing Toral Automorphisms*, M. Grayson, B. Kitchens, G. Zettler



$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^4$$

Images : *Visualizing Toral Automorphisms*, M. Grayson, B. Kitchens, G. Zettler

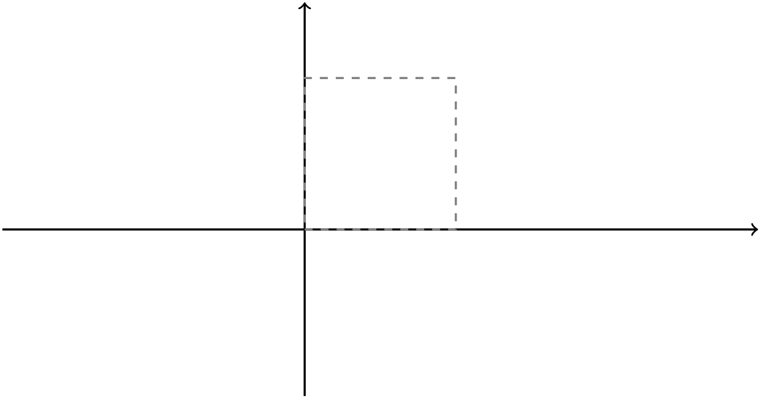


$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^5$$

Images : *Visualizing Toral Automorphisms*, M. Grayson, B. Kitchens, G. Zettler

Example 2 (detailed) : multiply by $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ in $[0,1]^2$

Dynamical system: $T \;:\; \begin{cases} [0,1]^2 & \to & [0,1]^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \;=\; \begin{pmatrix} x+y \\ x \end{pmatrix} \pmod 1 \end{cases}$



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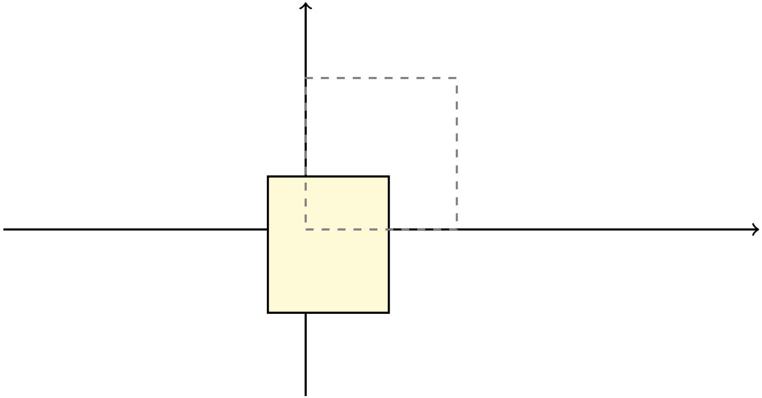
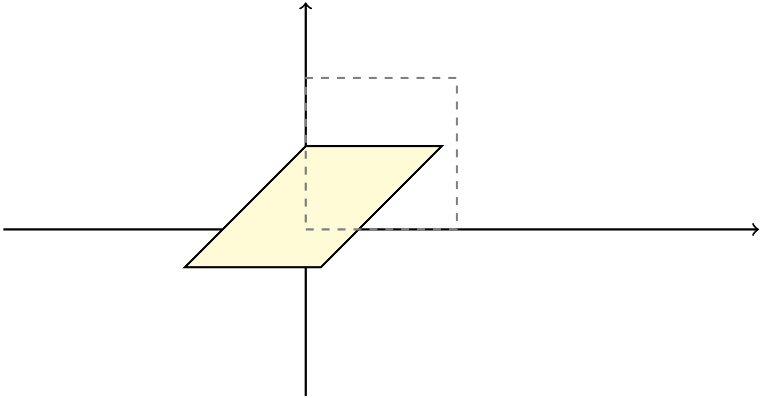


Example 2 (detailed) : multiply by $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ in $[0,1]^2$

Dynamical system: $T\ :\ \begin{cases} [0,1]^2 & \rightarrow & [0,1]^2 \\ \binom{x}{y} & \mapsto & \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)\binom{x}{y} \ = \ \binom{x+y}{x} \pmod 1 \end{cases}$



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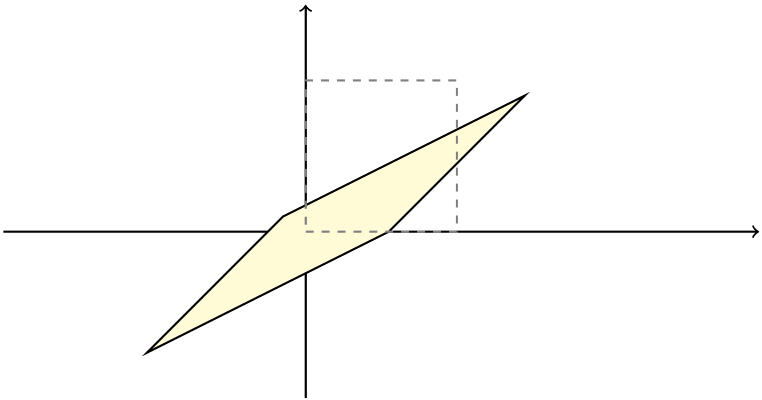

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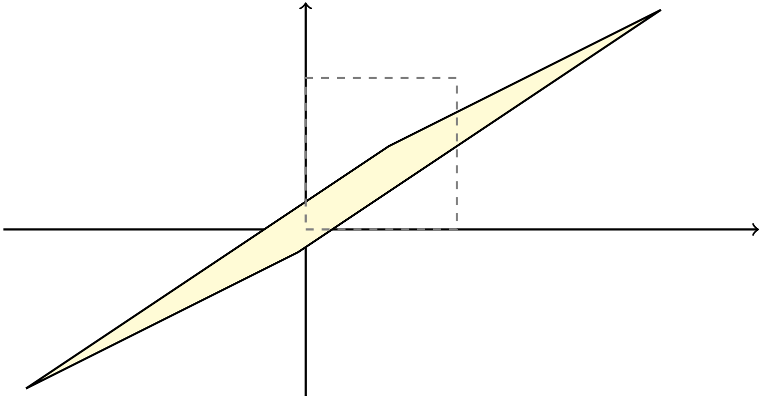

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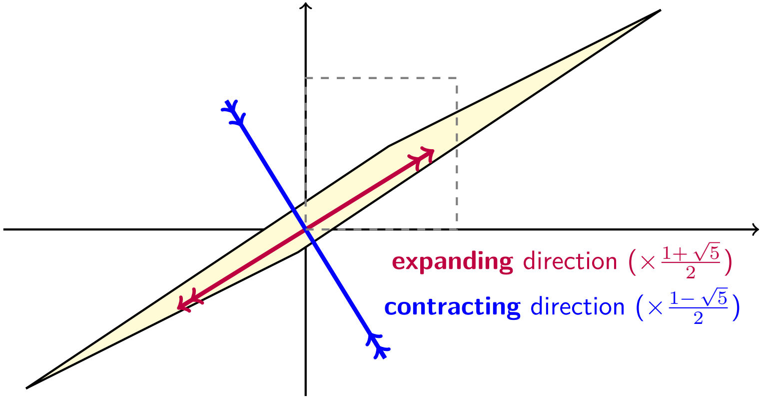
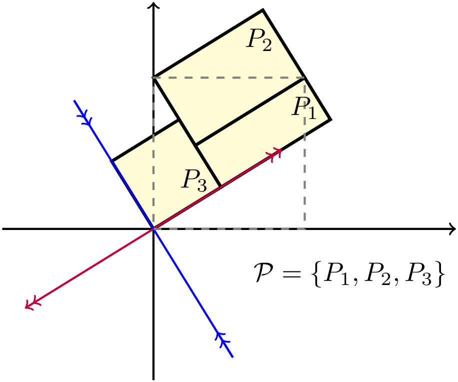
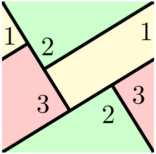


Choose a good partition $\mathcal{P}$ of $[0,1]^2$



$\mathcal{P} = \{P_1, P_2, P_3\}$

tore :



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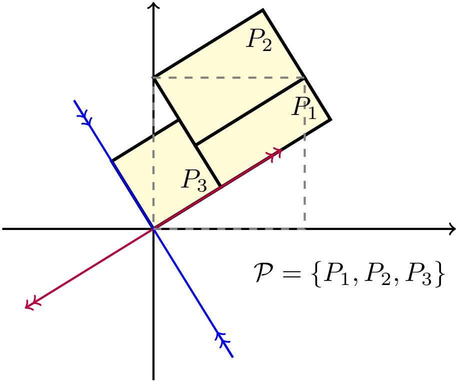

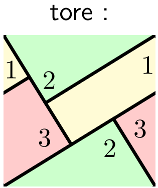

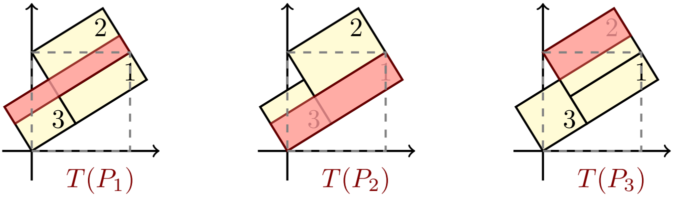

$\mathcal{P}$ is good

We want: To every sequence $(x_n) \in \Sigma_{\mathcal{P}}$ corresponds only one $x \in X$.

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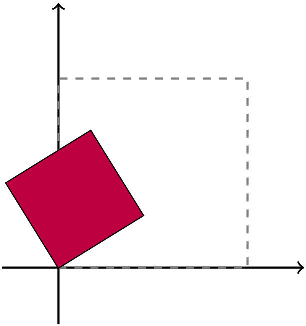
For example, for $(x_n) = \cdots 132\underline{3}2123 \cdots$ we have:



$\underline{3}$

P_3

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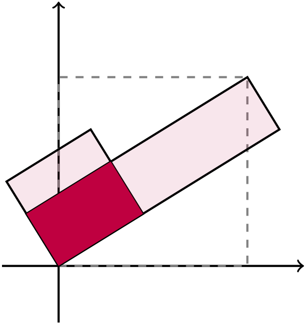

$2\underline{3}$

$T(P_2) \cap P_3$

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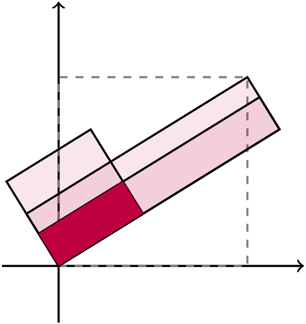

$32\underline{3}$

$T^2(P_3) \cap T(P_2) \cap P_3$

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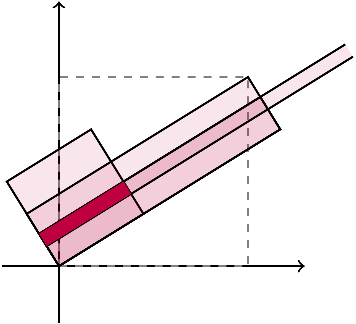

$132\underline{3}$

$T^3(P_1) \cap T^2(P_3) \cap T(P_2) \cap P_3$

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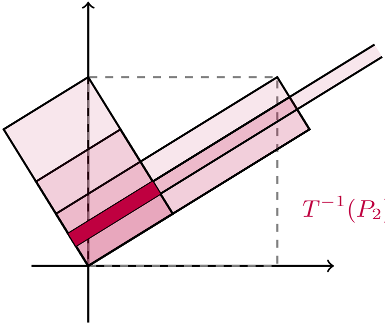

$1323\underline{3}2$

$T^3(P_1) \cap T^2(P_3) \cap T(P_2) \cap P_3 \cap$

$T^{-1}(P_2)$

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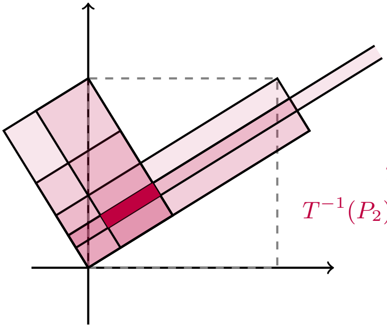

$132\underline{3}21$

$T^3(P_1) \cap T^2(P_3) \cap T(P_2) \cap P_3 \cap$
$T^{-1}(P_2) \cap T^{-2}(P_1)$

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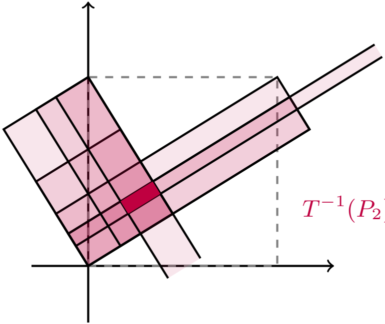

$132\underline{3}212$

$T^3(P_1) \cap T^2(P_3) \cap T(P_2) \cap P_3 \cap$
$T^{-1}(P_2) \cap T^{-2}(P_1) \cap T^{-3}(P_2)$

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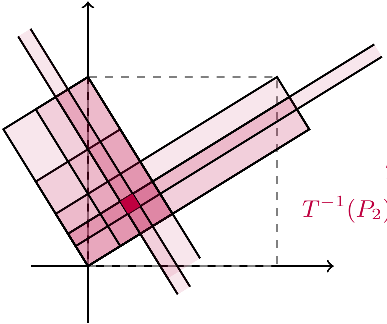

$132\underline{3}2123$

$$T^3(P_1) \cap T^2(P_3) \cap T(P_2) \cap P_3 \cap$$
$$T^{-1}(P_2) \cap T^{-2}(P_1) \cap T^{-3}(P_2) \cap T^{-3}(P_3)$$

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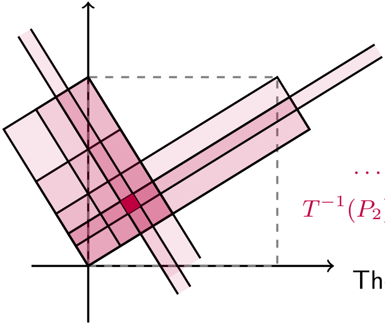

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The limit contains only one point!

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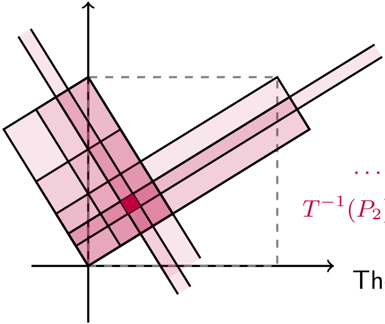

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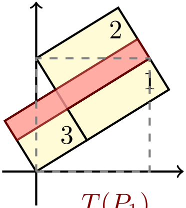
The limit contains only one point!

$(\Sigma_{\mathcal{P}}, \text{shift})$ is then a **good representation** of (X, T).

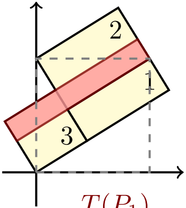
Precise description of $\Sigma_{\mathcal{P}}$



$T(P_1)$

admissible : $12, 13$

Precise description of $\Sigma_{\mathcal{P}}$



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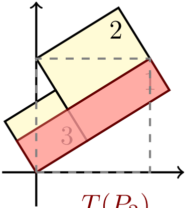
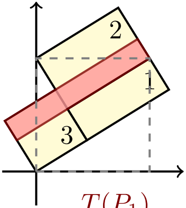


$T(P_2)$

admissible : $21, 23$

Precise description of $\Sigma_{\mathcal{P}}$



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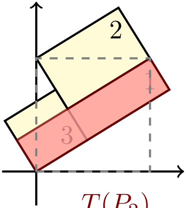

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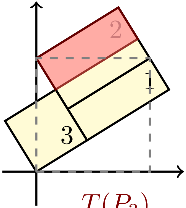
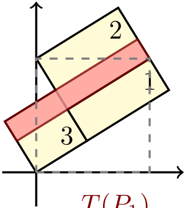


$T(P_3)$

admissible : 32

Precise description of $\Sigma_{\mathcal{P}}$



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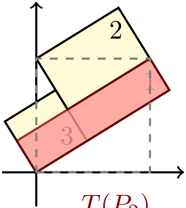

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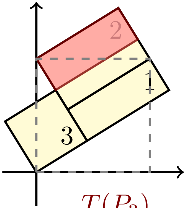

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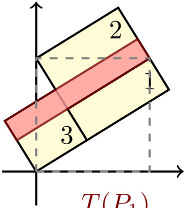
admissible : 32

- Geometrical inspection: ij and jk admissible $\Longrightarrow$ ijk admissible
  ➥ k does not depend on i in ijk (Markov property)

Precise description of $\Sigma_{\mathcal{P}}$



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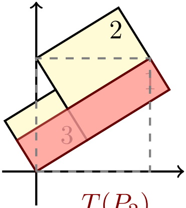
admissible : $12, 13$



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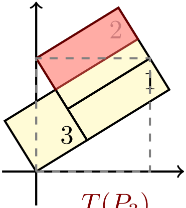

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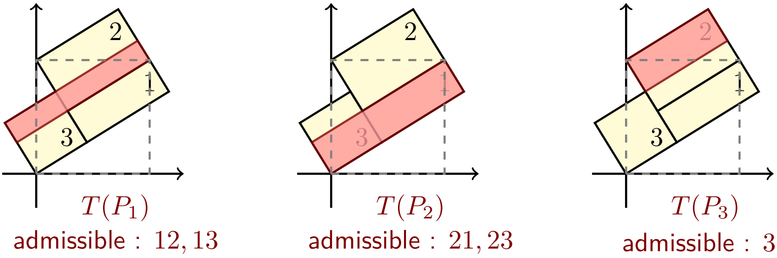
So: $\Sigma_{\mathcal{P}} = \{$sequences avoiding $11, 22, 31, 33\} \subseteq \{1, 2, 3\}^{\mathbb{Z}}$.

Precise description of $\Sigma_{\mathcal{P}}$



- Geometrical inspection: ij and jk admissible $\implies$ ijk admissible
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So: $\Sigma_{\mathcal{P}} = \{$sequences avoiding $11, 22, 31, 33\} \subseteq \{1, 2, 3\}^{\mathbb{Z}}$.

➥ $(\Sigma_{\mathcal{P}}, \text{shift})$ is then a subshift of finite type

➥ $\mathcal{P}$ is a Markov partition

Consequences

- Periodic points of $\Sigma_{\mathcal{P}}$ are dense, so those of $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are too.
- Entropy can easily be computed.
- $\Sigma_{\mathcal{P}}$ is transitive so $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ too.
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We have a general result:

Theorem [Berg 67, Adler-Weiss 67]

There exists a Markov partition for every 2×2 matrix $M \in \mathcal{M}_2(\mathbb{Z})$ s.t.:

- $\det(M) = \pm 1$;
- M is hyperbolic (one expanding e.v., one contracting e.v.).

➥ Explicit construction with two rectangles!

What about. . .

dimension $\geqslant 3$?

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Theorem ☺[Sinaĭ 68, Bowen 70's]

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➥ "Optimal" : $\exists$ e.v. of modulus $1 \Longrightarrow$ no Markov partition [Lind 78]

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- ➥ "Optimal" : $\exists$ e.v. of modulus $1 \Longrightarrow$ no Markov partition [Lind 78]
- ➥ **But:** non-explicit construction...

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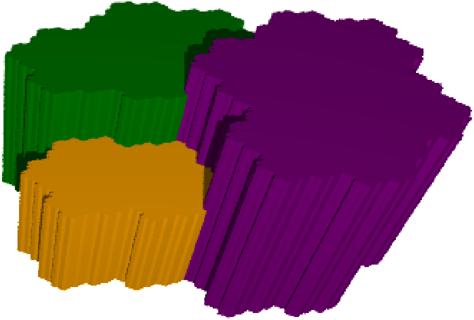

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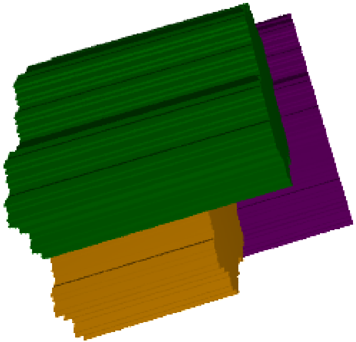

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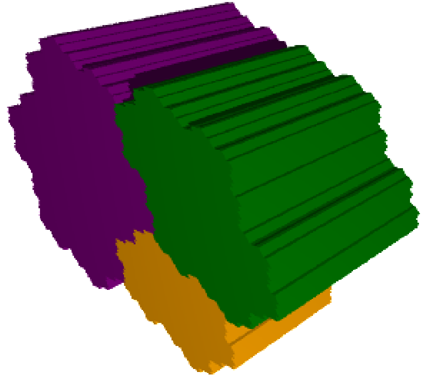

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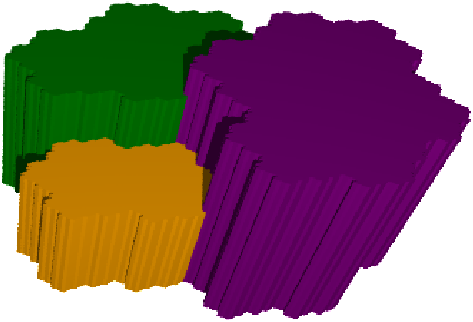

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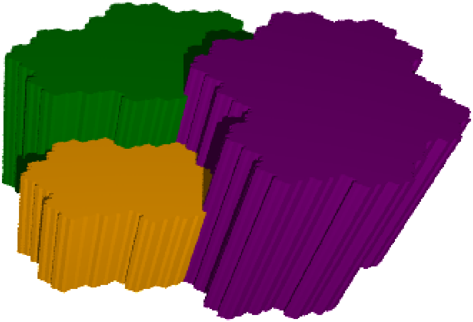

- We need **new tools** to explain further.

Tools:

- substitutions
- Rauzy fractals

Tools: substitutions

Substitution:

$$\sigma : \begin{cases} 1 & \mapsto & 12 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 1 \end{cases} \qquad\qquad \mathbf{M}_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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σ is **Pisot**: (special case of hyperbolic)

- **Exactly one** expanding e.v. β
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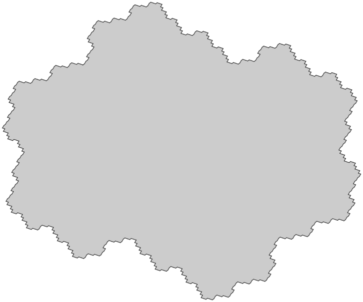
Action of $\mathbf{M}_\sigma$ on $\mathbb{R}^3$:

- A **expanding line** spanned by eigenvector v_β
- A **contractanting plane** spanned by the two other eigenvectors

Tools: Rauzy fractals

In the contracting plane of $\mathbf{M}_\sigma$ lives **Rauzy Fractal** of σ.
[Rauzy 1982, Arnoux-Ito 2001]



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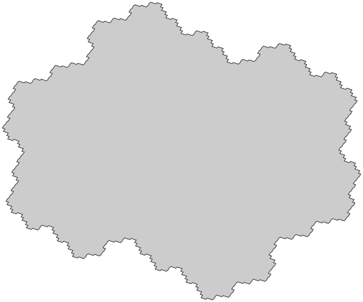

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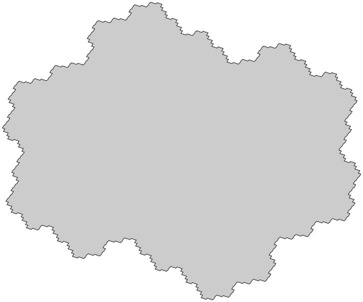

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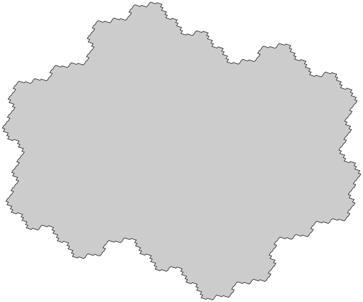

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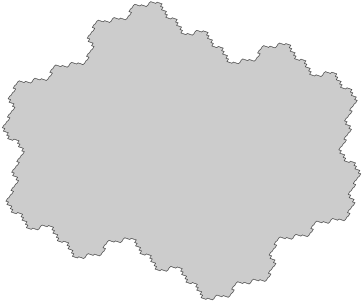

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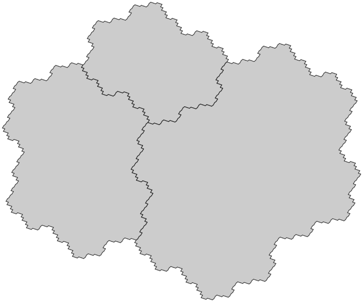

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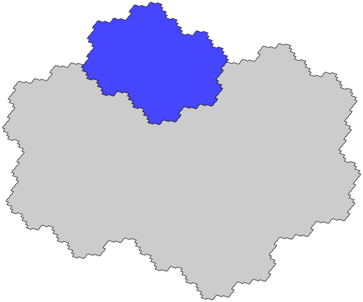

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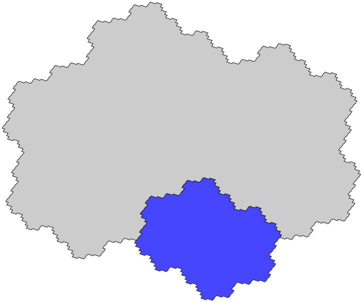

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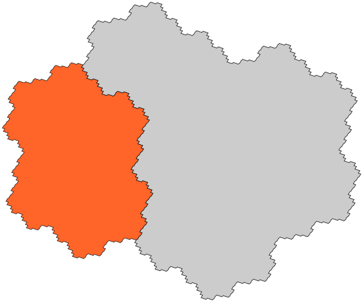

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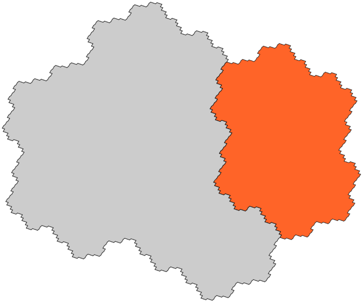

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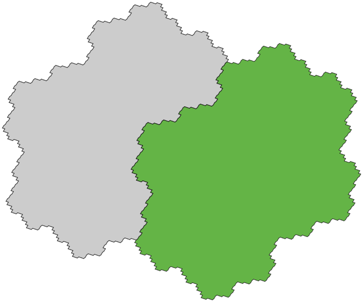

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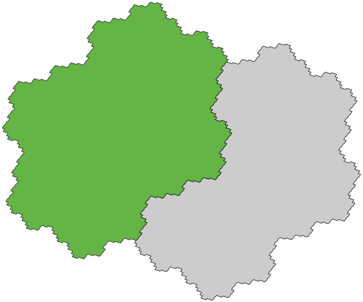

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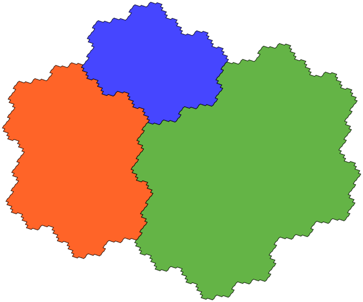

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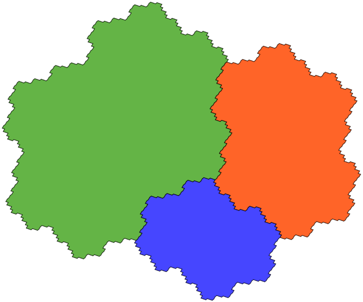

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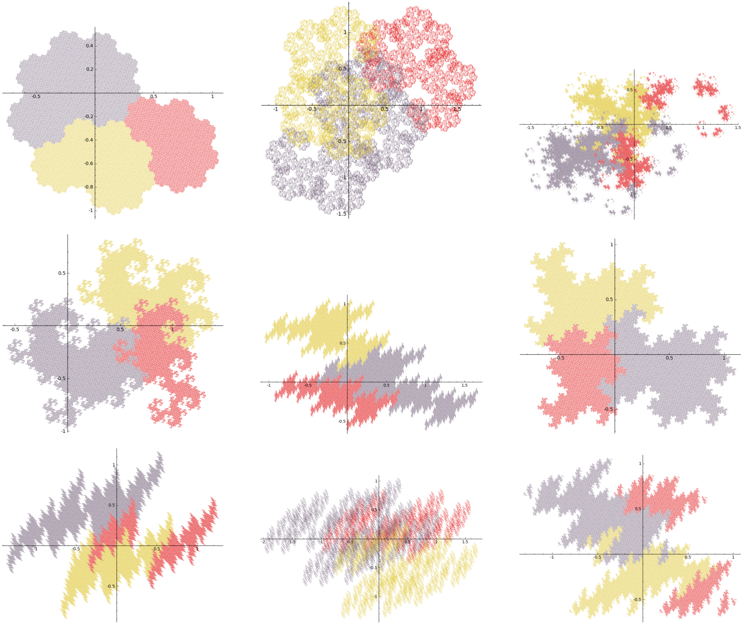

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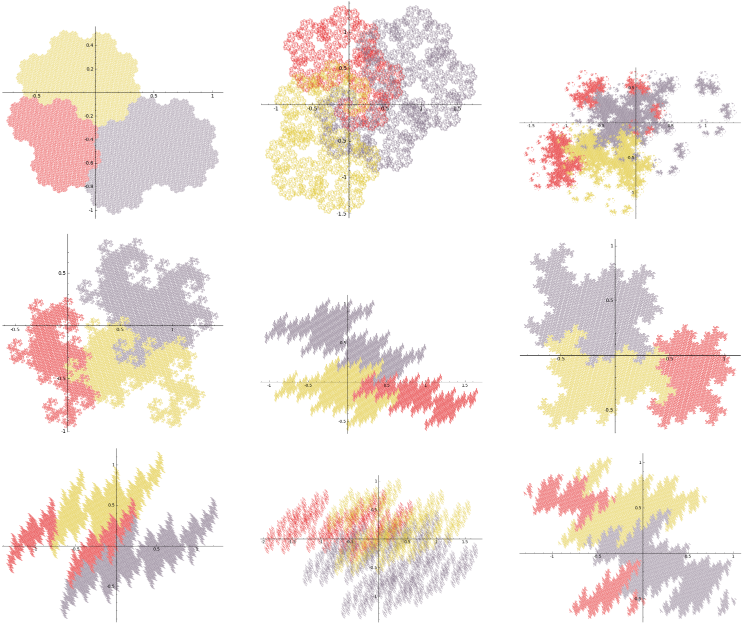

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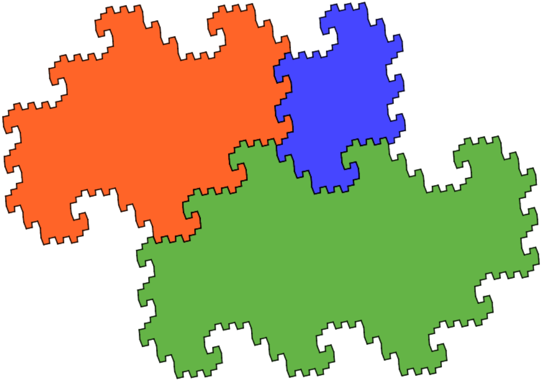
- ...very different from subshifts of finite type $\Sigma_{\mathcal{P}}$:
  - minimal sytem,
  - zero entropy,
  - no periodic points...

Properties: dynamics of (X_σ, shift), geometrically



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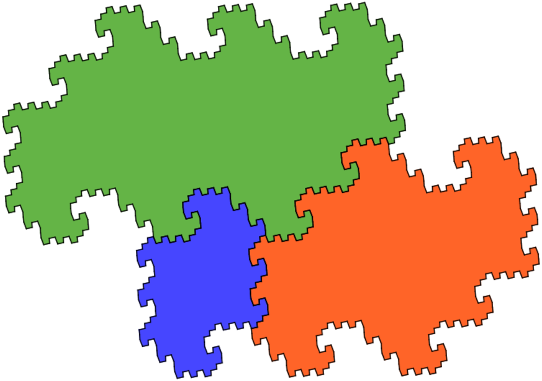

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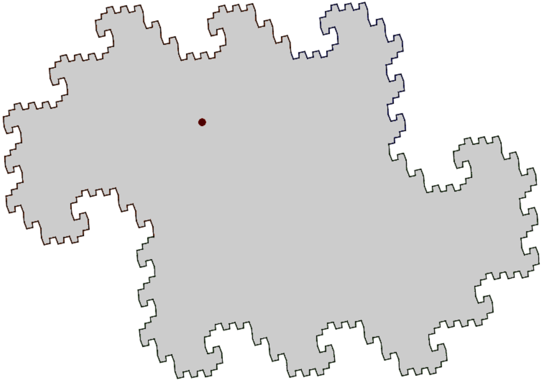

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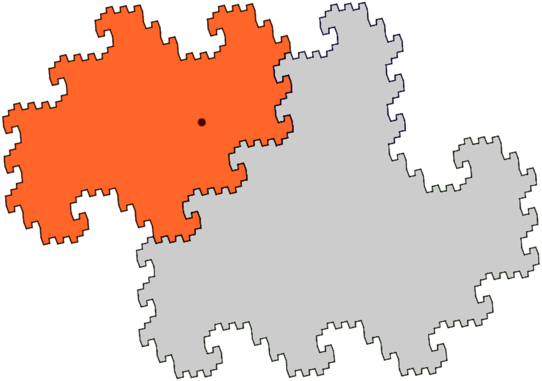

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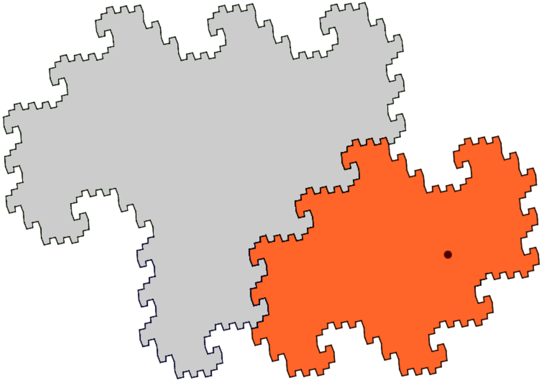

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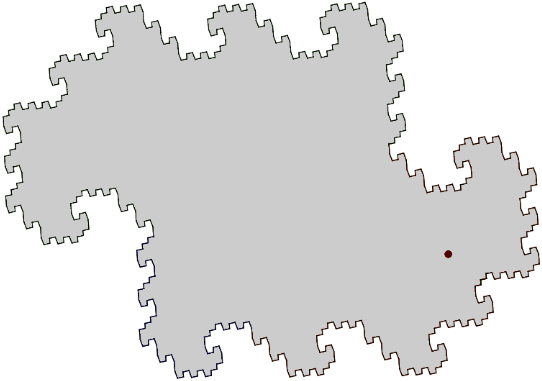

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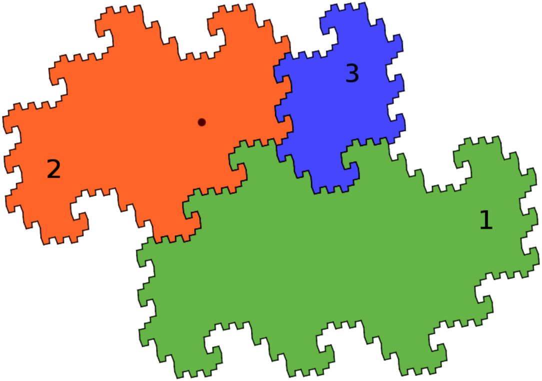
Orbit : 2



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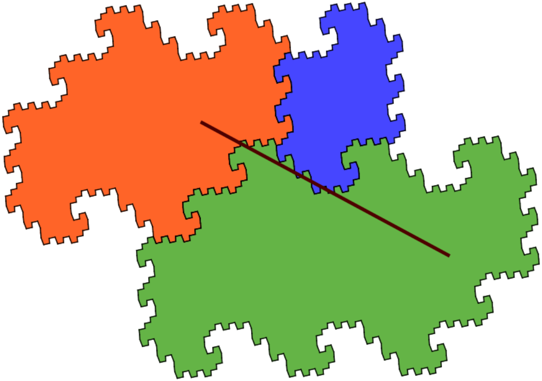

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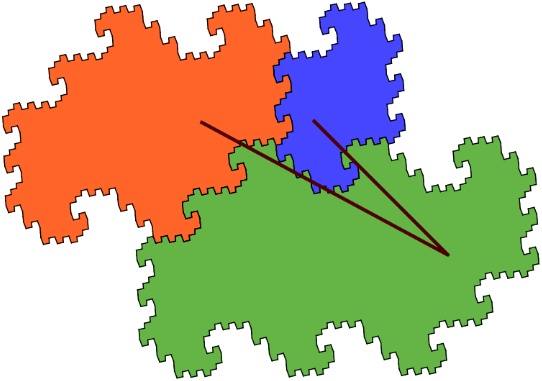

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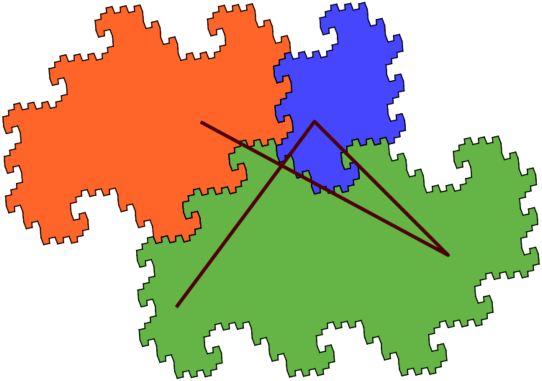

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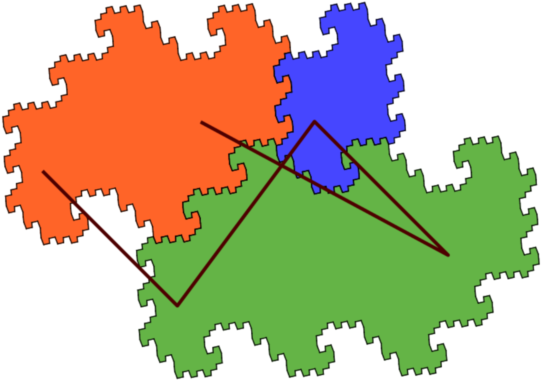

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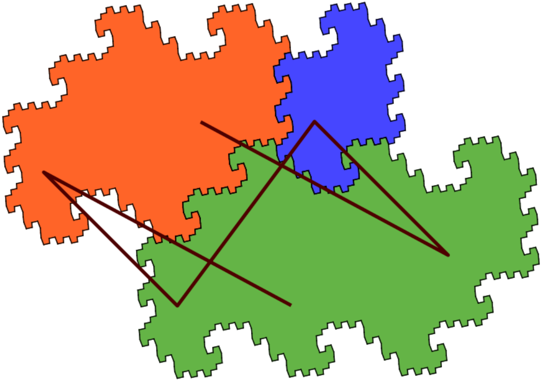

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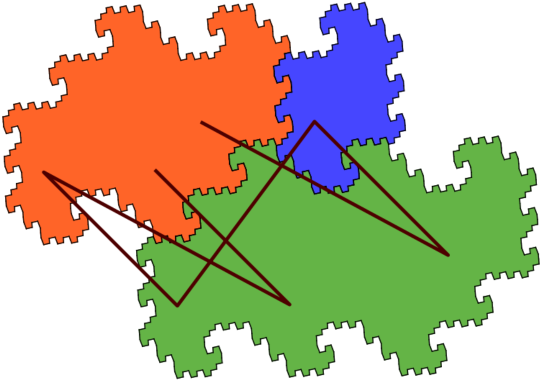

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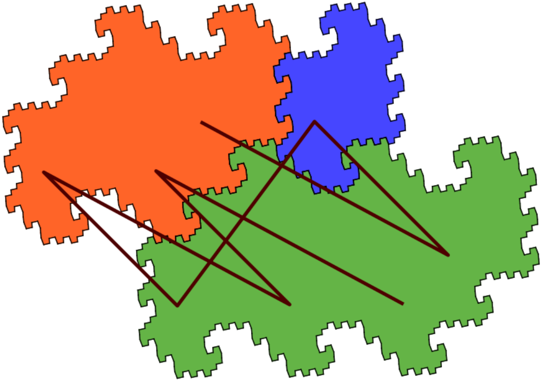

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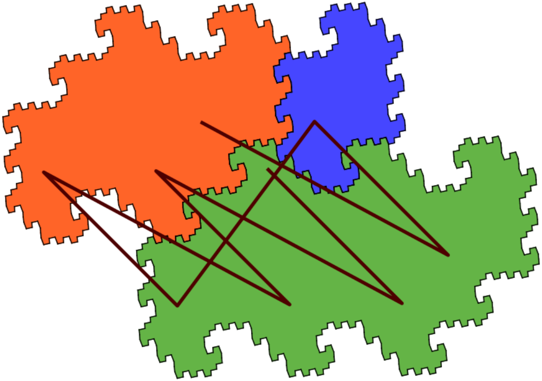

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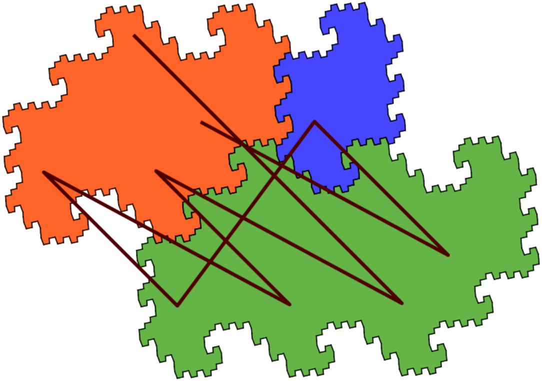

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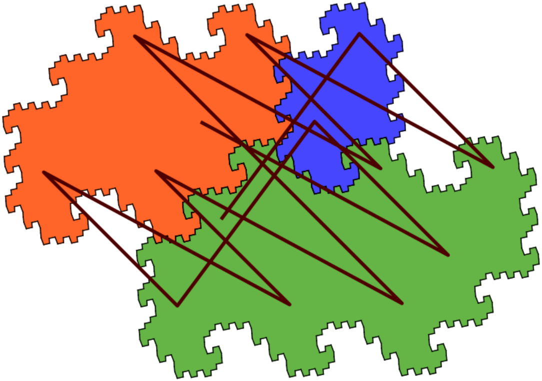

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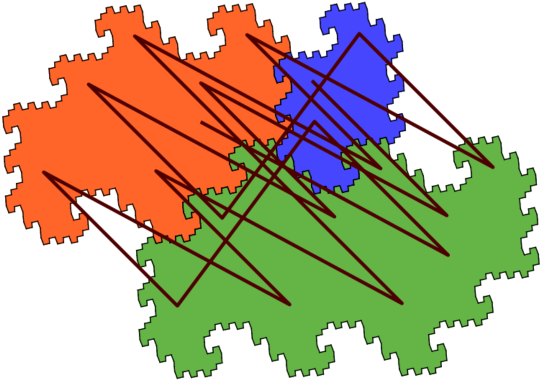

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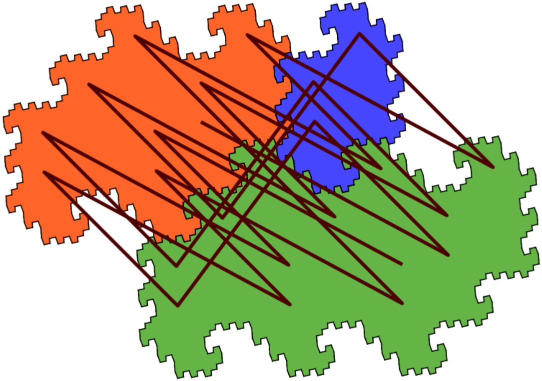

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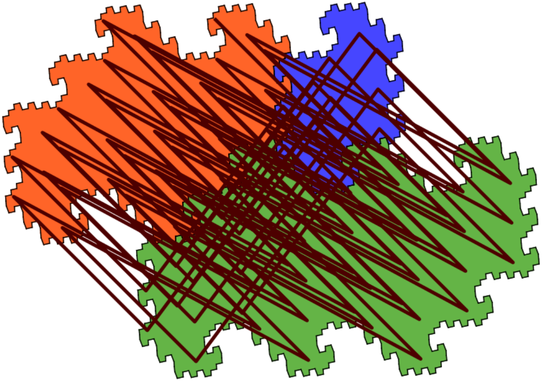
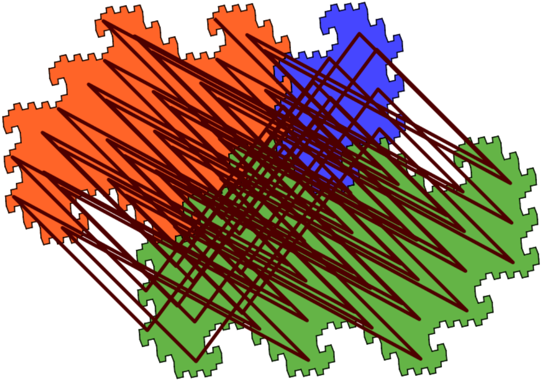


Properties: dynamics of (X_σ, shift), geometrically

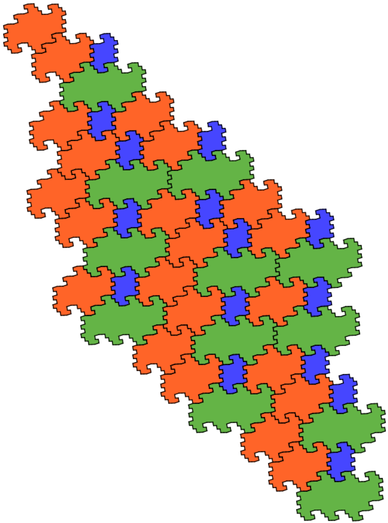
Orbit : $\cdots 2131212112121312121312121 \cdots \in X_\sigma \subseteq \{1,2,3\}^{\mathbb{Z}}$



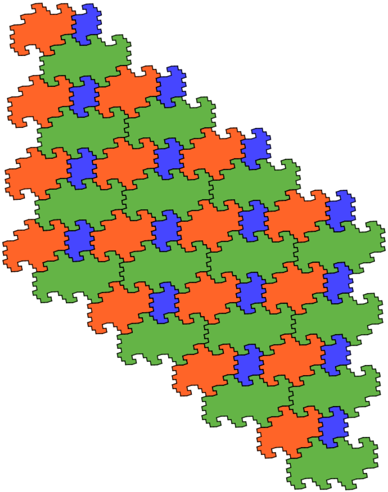
$\Big(X_\sigma\,,\ \text{shift}\Big) \quad \cong \quad \Big($  , exchange$\Big)$

Properties: tilings

Self-similar tiling (aperiodic) :

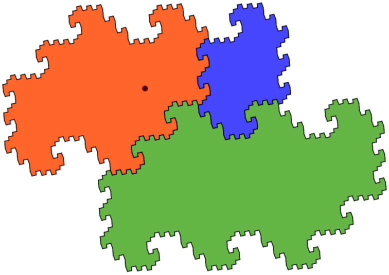


Periodic tiling :

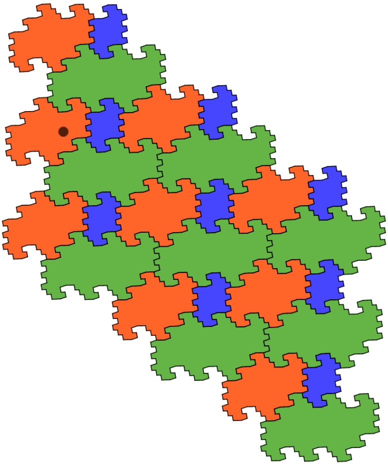


Properties: dynamics of σ, continued

(1) Domain exchange:



(2) Translation on the torus $\mathbb{T}^2$:

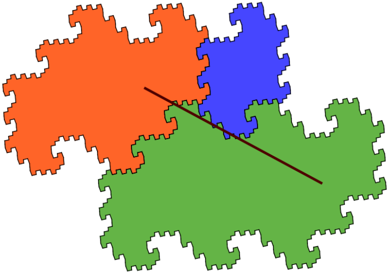


(3) Shift:

$$\cdots \underline{2}131212112 \cdots \in X_\sigma$$

Properties: dynamics of σ, continued

(1) Domain exchange:



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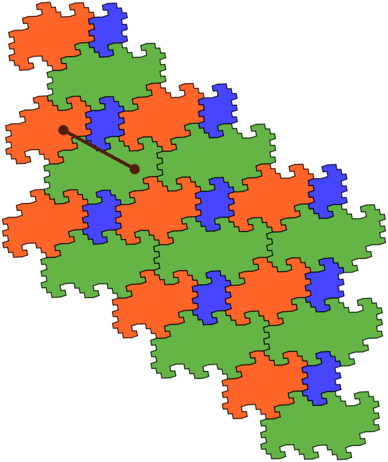
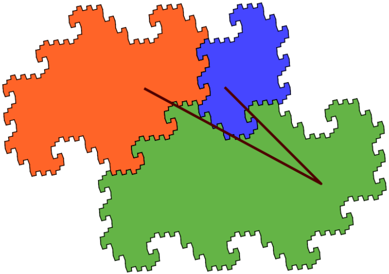


(3) Shift:

$$\cdots 2\underline{1}31212112 \cdots \in X_\sigma$$

Properties: dynamics of σ, continued

(1) Domain exchange:



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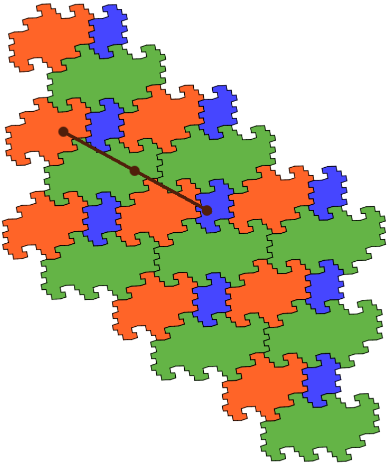
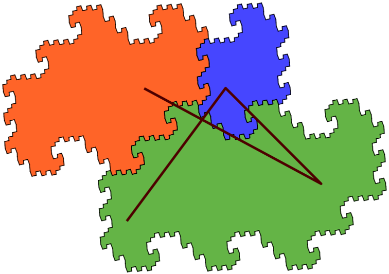


(3) Shift:

$$\cdots 21\underline{3}1212112 \cdots \in X_\sigma$$

Properties: dynamics of σ, continued

(1) Domain exchange:



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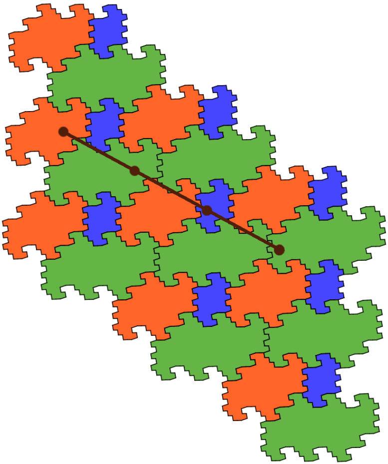
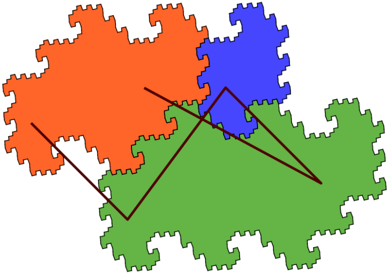


(3) Shift:

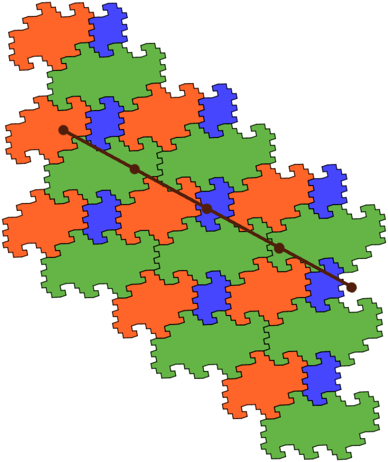
$$\cdots 2131\underline{1}212112 \cdots \in X_\sigma$$

Properties: dynamics of σ, continued

(1) Domain exchange:



(2) Translation on the torus $\mathbb{T}^2$:

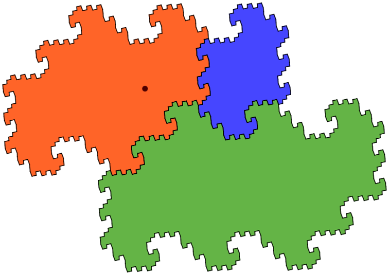


(3) Shift:

$$\cdots 2131\underline{2}12112 \cdots \in X_\sigma$$

Properties: dynamics of σ, continued

(1) Domain exchange:



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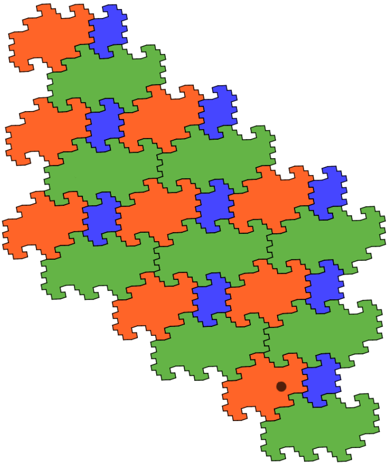

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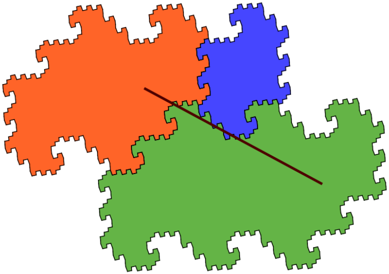
Properties: dynamics of σ, continued

(1) Domain exchange:



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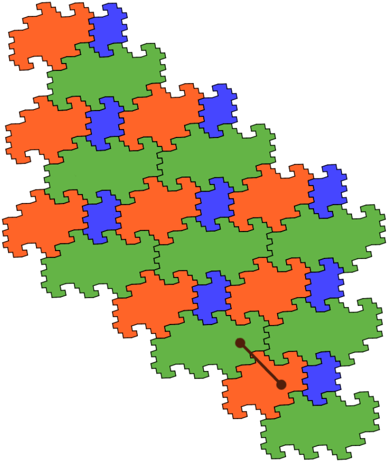

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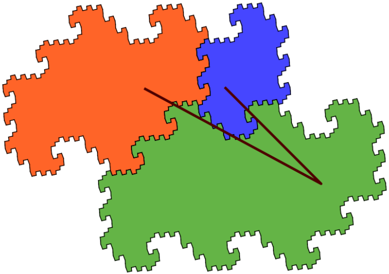
Properties: dynamics of σ, continued

(1) Domain exchange:



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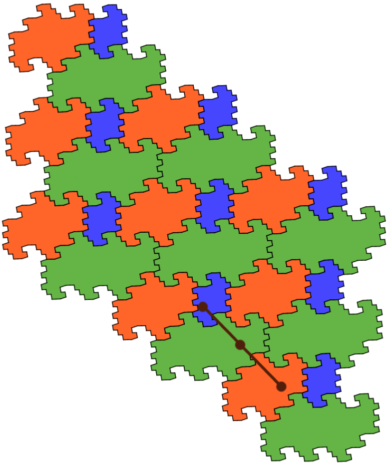
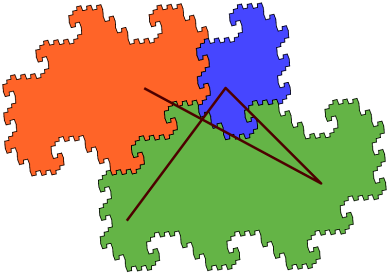


(3) Shift:

$$\cdots 21\underline{3}1212112 \cdots \in X_\sigma$$

Properties: dynamics of σ, continued

(1) Domain exchange:



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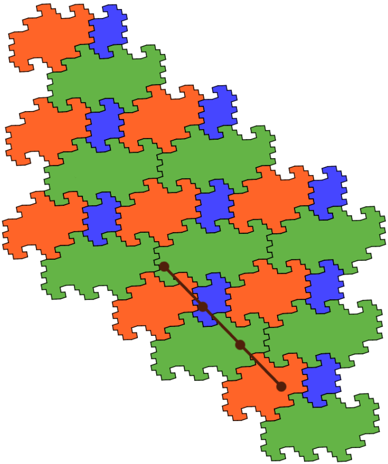

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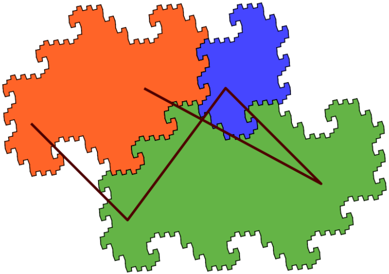
Properties: dynamics of σ, continued

(1) Domain exchange:



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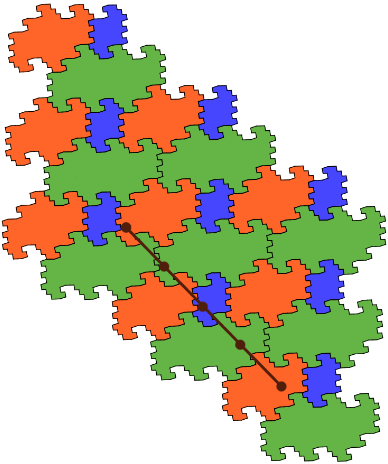
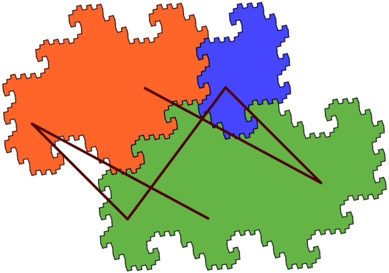


(3) Shift:

$$\cdots 2131\underline{2}12112 \cdots \in X_\sigma$$

Properties: dynamics of σ, continued

(1) Domain exchange:



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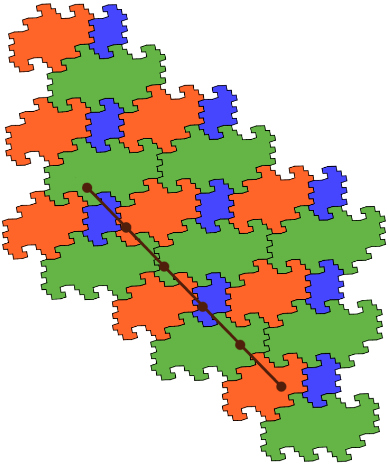
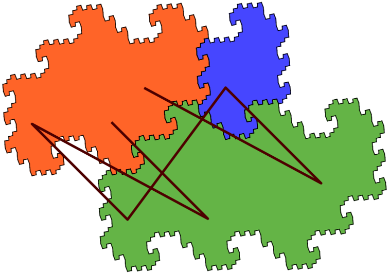


(3) Shift:

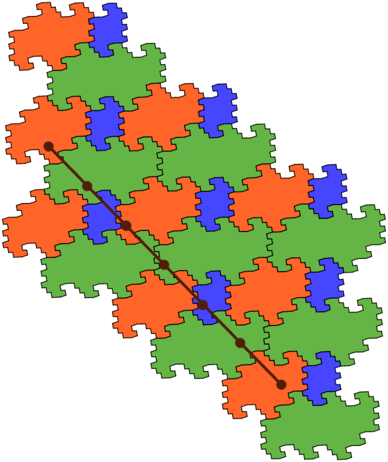
$$\cdots 21312\underline{1}2112 \cdots \in X_\sigma$$

Properties: dynamics of σ, continued

(1) Domain exchange:



(2) Translation on the torus $\mathbb{T}^2$:

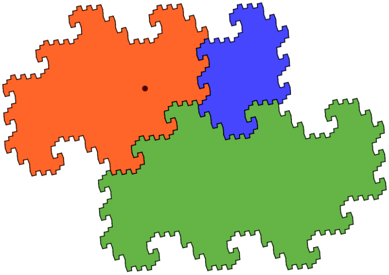


(3) Shift:

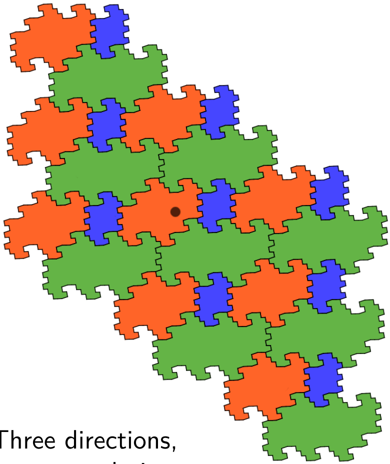
$$\cdots 213121\underline{2}112 \cdots \in X_\sigma$$

Properties: dynamics of σ, continued

(1) Domain exchange:



(2) Translation on the torus $\mathbb{T}^2$:



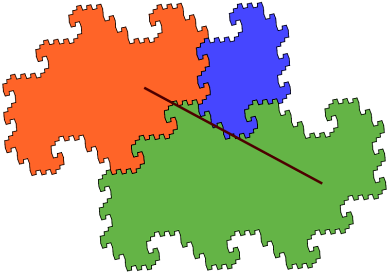
Three directions,
same translation

(3) Shift:

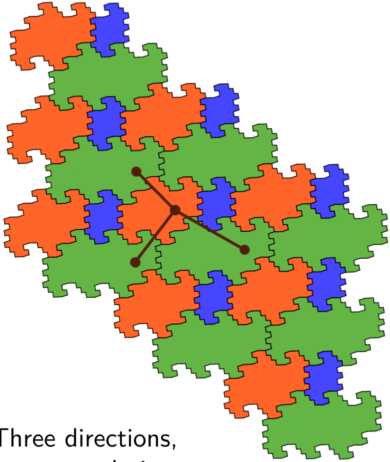
$$\cdots \underline{2}131212112 \cdots \in X_\sigma$$

Properties: dynamics of σ, continued

(1) Domain exchange:



(2) Translation on the torus $\mathbb{T}^2$:



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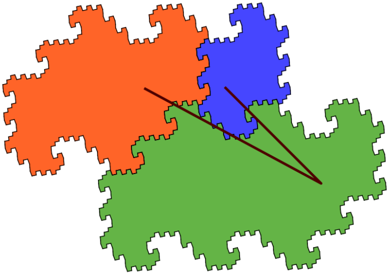
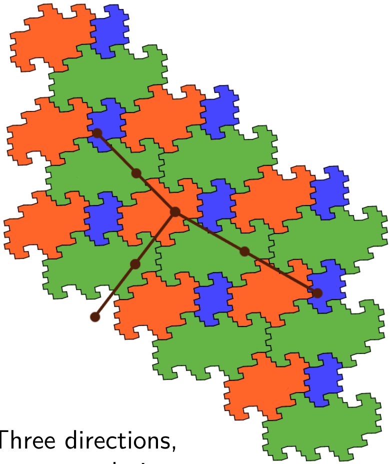
Properties: dynamics of σ, continued

(1) Domain exchange:



(2) Translation on the torus $\mathbb{T}^2$:



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Properties: dynamics of σ, continued

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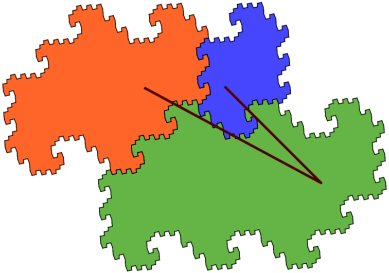

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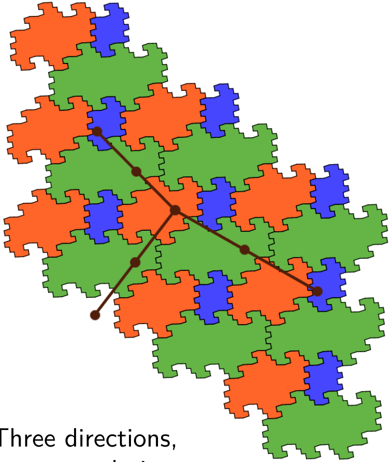

Three directions,
same translation

(3) Shift:

$$\cdots 21\underline{3}1212112 \cdots \ \in X_\sigma$$

$$\textbf{(1)} \iff \textbf{(2)} \iff \textbf{(3)}$$

Properties: dynamics of σ, continued

- $(X_\sigma, \text{shift}) \cong ($  $, \text{exchange}) \cong (\mathbb{T}^2, \text{translation})$

Properties: dynamics of σ, continued

- $\left(X_\sigma\,, \text{shift}\right) \;\cong\; \Big($  $, \text{exchange}\Big) \;\cong\; \left(\mathbb{T}^2\,, \text{translation}\right)$

  ➥ Other example of symbolic coding (with partition )

Properties: dynamics of σ, continued

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    ➥ Other example of symbolic coding (with partition )

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Properties: dynamics of σ, continued

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  - ➥ Other example of symbolic coding (with partition )
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- **Back to the action of** $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ **on** $\mathbb{T}^3$

Properties: dynamics of σ, continued

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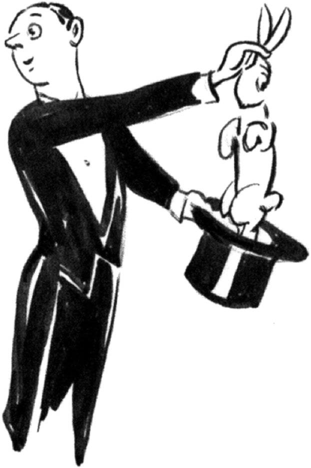
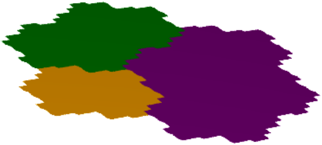
- **Back to the action of** $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ **on** $\mathbb{T}^3$

- **We now build a Markov partition for** $(\mathbb{T}^3, \mathbf{M}_\sigma)$



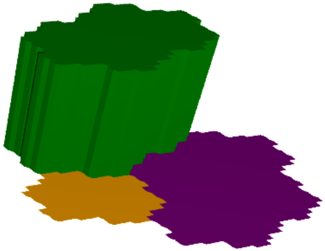
Markov partition for $(\mathbb{T}^3, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix})$

1. Take the Rauzy fractal in the contracting plane



Markov partition for $\left(\mathbb{T}^3, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right)$

1. Take the Rauzy fractal in the contracting plane
2. "Suspend" each domain in the expanding direction



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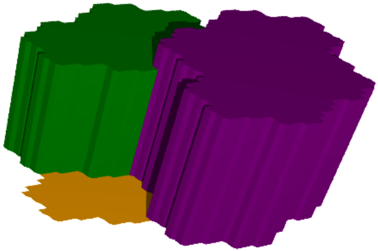

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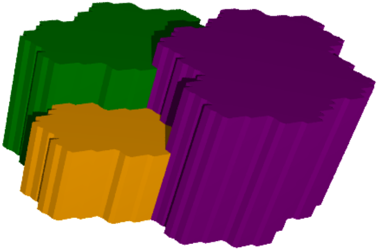

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1. Take the Rauzy fractal in the contracting plane
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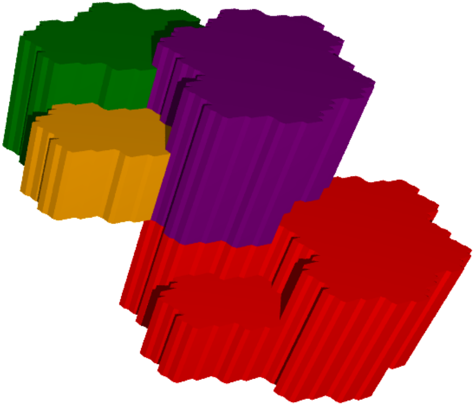
3. It tiles periodically! — Fundamental domain of the torus $\mathbb{T}^3$



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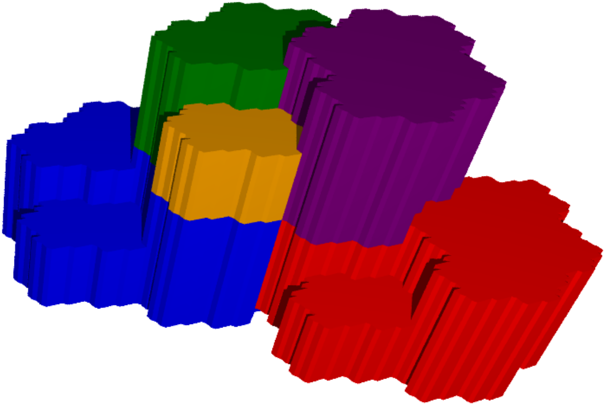

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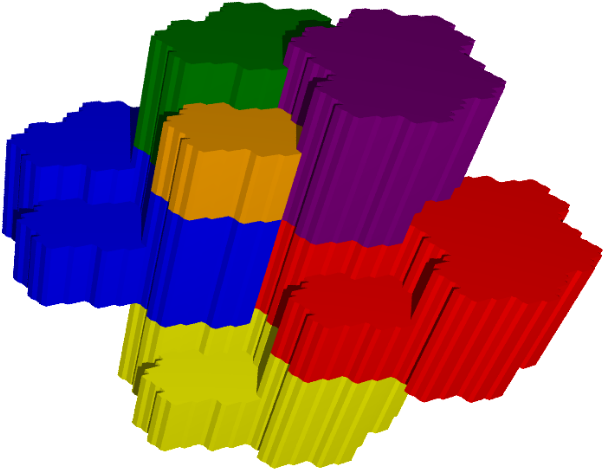

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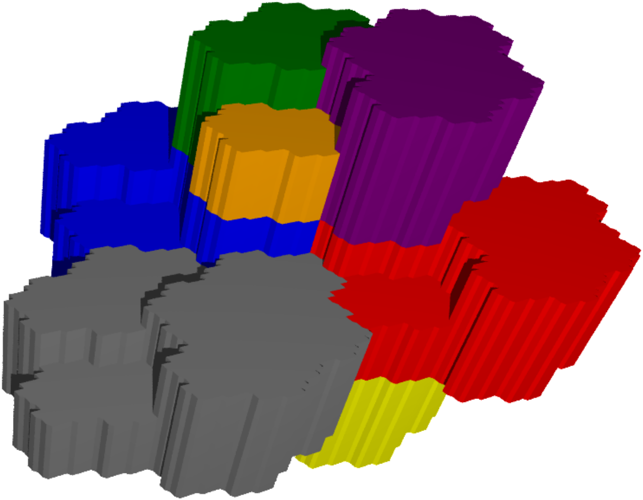

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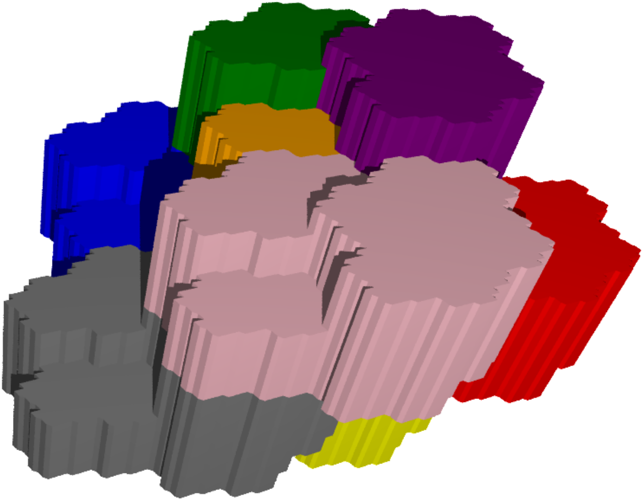

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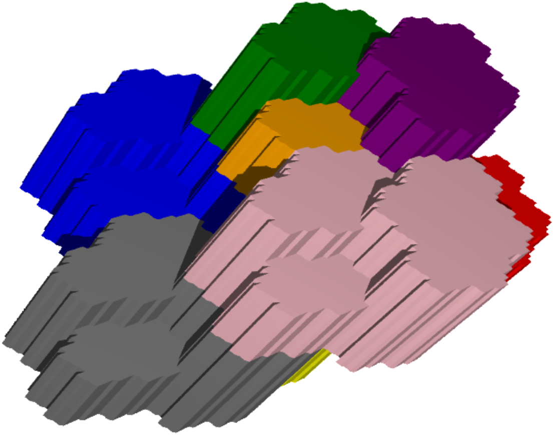

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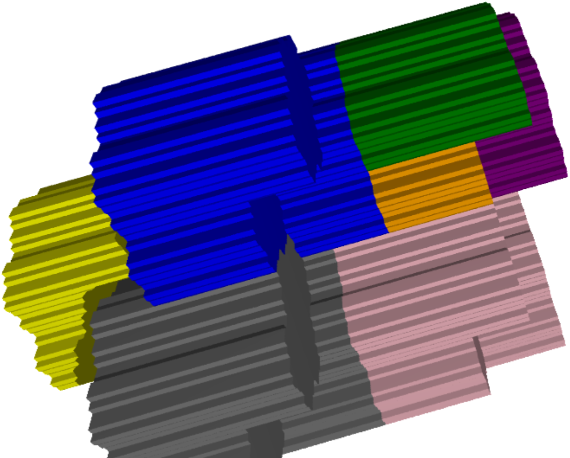

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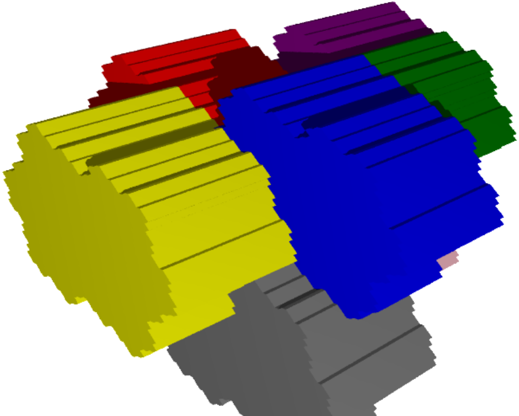

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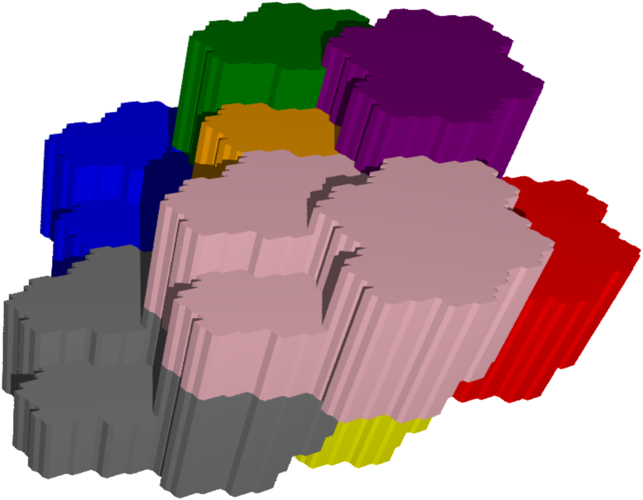

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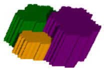
The tiling follows a 3D-lattice:



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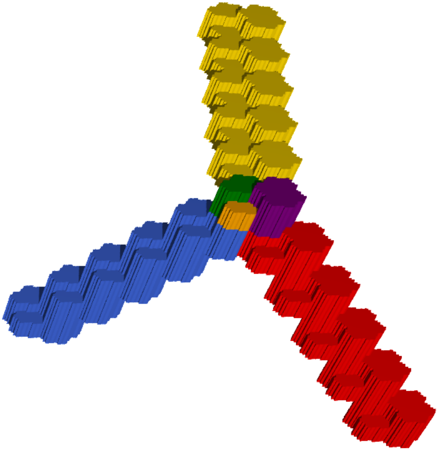

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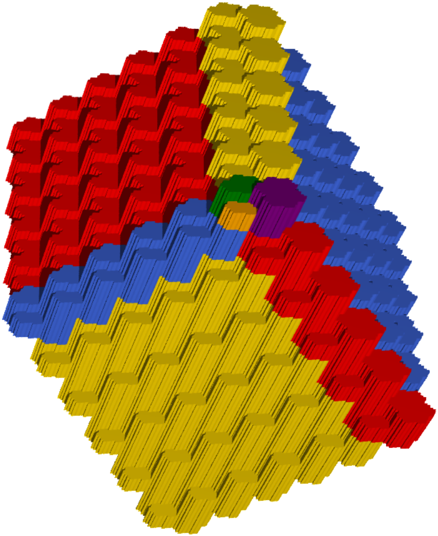

Partition de Markov for $(\mathbb{T}^3, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix})$

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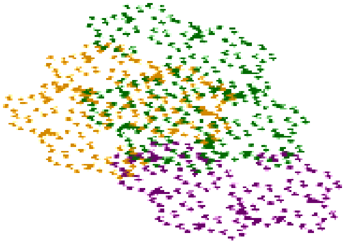

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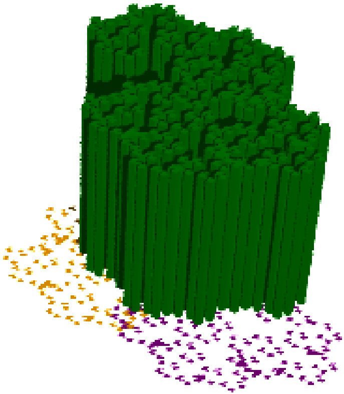

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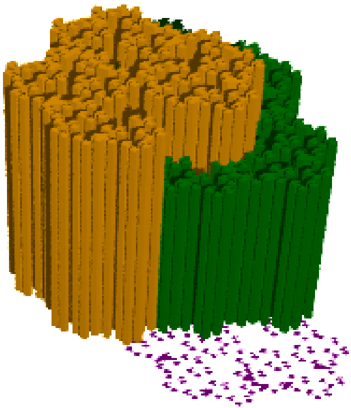

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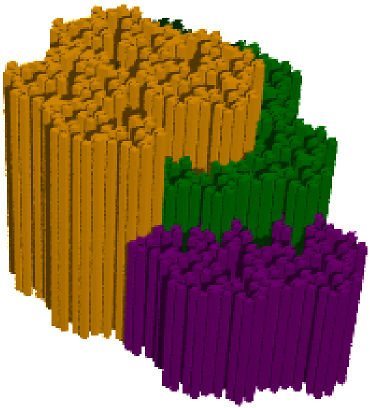

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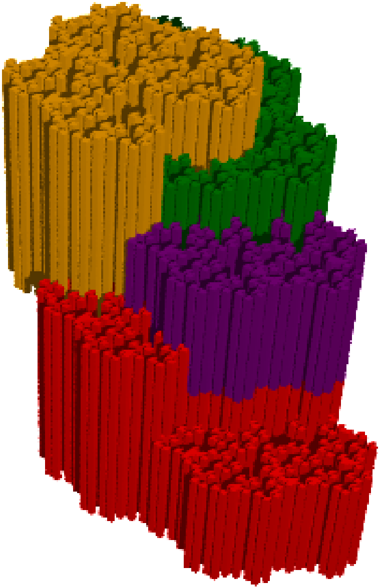

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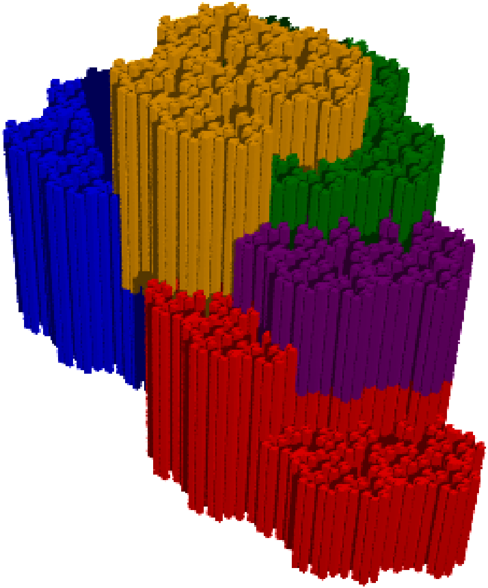

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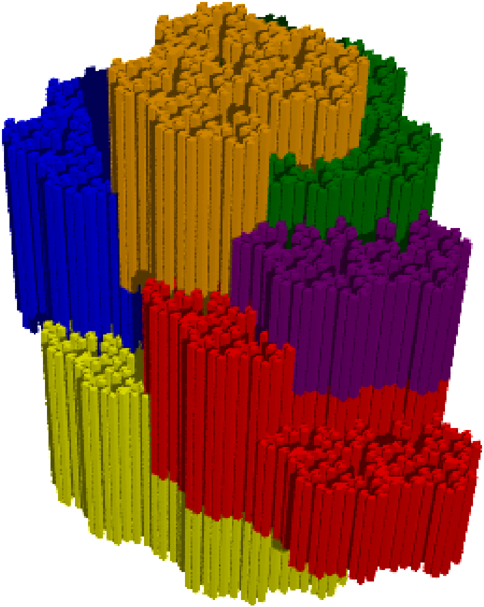

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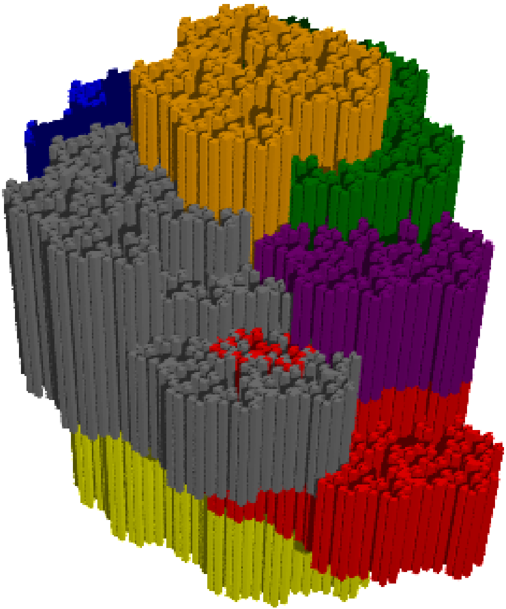
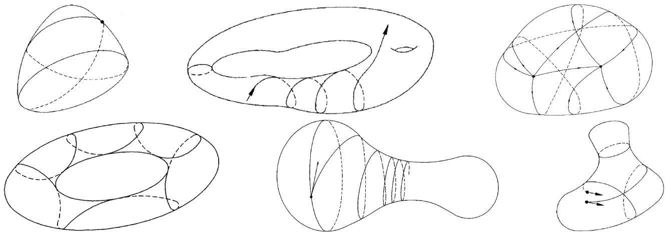


谢谢大家的关注。



- **The Sage software**, `www.sagemath.org`
  Used here to draw pictures; a free and very complete math software.
- **R. L. Adler**, *Symbolic Dynamics and Markov Partitions*, Bulletin of the AMS, 1998. For automorphisms of $\mathbb{T}^2$.
- **Anne Siegel**, HDR (online). For $\mathbb{T}^d$, $d \geqslant 3\ldots$