Connectedness of Rauzy fractal families

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Joint work with Valérie Berthé and Anne Siegel

Substitutive Tilings and Fractal Geometry
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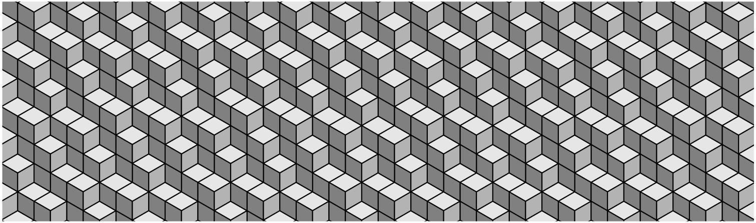
Discrete planes

Let $v = (a, b, c) \in \mathbb{R}^3_+$ not equal to $(0, 0, 0)$.

Definition: discrete plane $\mathcal{P}_{\mathbf{v}}$ of normal vector v

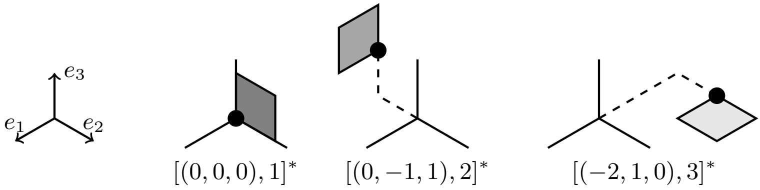
$\mathcal{P}_{\mathbf{v}}$ = the boundary of the union of the unit cubes that intersect the half-space defined by $\langle x, v \rangle < 0$



Unit faces

Every discrete plane is covered by **unit faces**, denoted $[\mathbf{x}, i]^*$:

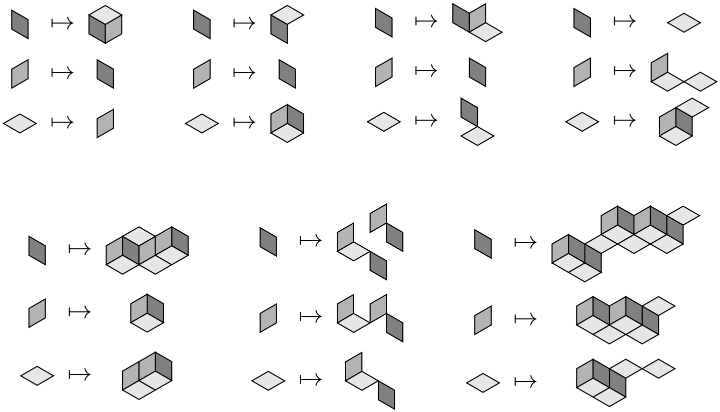
- **position** $\mathbf{x} \in \mathbb{Z}^3$
- **type** $i \in \{1, 2, 3\}$



Notation: $\mathbf{x} + D$: translate a union of faces D by $\mathbf{x} \in \mathbb{Z}^3$

Now, let's play with lozenges. . .

Some lozenge substitutions



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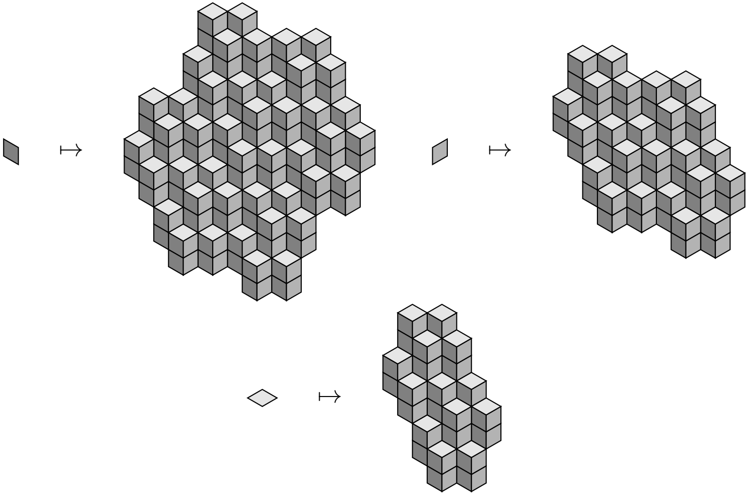

...these substitutions all come from:

Definition [Arnoux-Ito '01]

$$\mathbf{E}_1^*(\sigma)([\mathbf{x},i]^*) \;=\; \bigcup_{k=1,2,3}\;\bigcup_{s\,|\,\sigma(k)=pis} [\mathbf{M}_\sigma^{-1}(x+\ell(s)),k]^*,$$

where:

- $\sigma : \{1,2,3\}^* \to \{1,2,3\}^*$ is a **unimodular** substitution
- $\mathbf{M}_\sigma$ is the incidence matrix of σ
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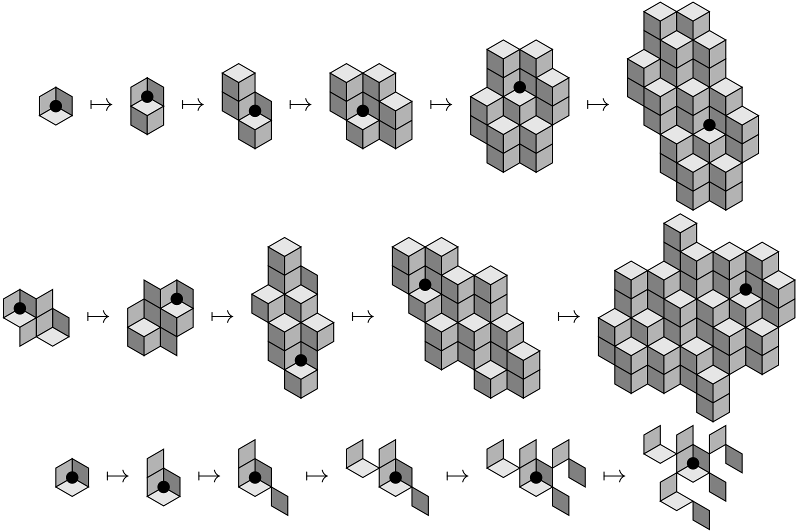
Example for $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$

$$\begin{aligned}
\mathbf{E}_1^*(\sigma)([\mathbf{x}, 1]^*) &= \mathbf{M}_\sigma^{-1}\mathbf{x} + [(1,0,-1),1]^* \cup [(0,1,-1),2]^* \cup [(0,0,0),3]^* \\
\mathbf{E}_1^*(\sigma)([\mathbf{x}, 2]^*) &= \mathbf{M}_\sigma^{-1}\mathbf{x} + [(0,0,0),1]^* \\
\mathbf{E}_1^*(\sigma)([\mathbf{x}, 3]^*) &= \mathbf{M}_\sigma^{-1}\mathbf{x} + [(0,0,0),2]^*
\end{aligned}$$

Some examples



Some properties of $\mathbf{E}_1^*(\sigma)$

Miracle 1: images do not overlap [Arnoux-Ito '01]

$$[\mathbf{x}, i]^* \neq [\mathbf{x}', i']^* \in \mathcal{P}_{\mathbf{v}} \quad \Longrightarrow \quad \mathbf{E}_1^*(\sigma)([\mathbf{x}, i]^*) \cap \mathbf{E}_1^*(\sigma)([\mathbf{x}', i']^*) = \varnothing$$

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Linearity (Arnoux-Ito '01)

$$\mathbf{E}_1^*(\sigma)([\mathbf{x}, i]^*) \;=\; \mathbf{M}_\sigma^{-1}\mathbf{x} + \mathbf{E}_1^*(\sigma)([(0,0,0), i]^*)$$

➥ $\mathbf{E}_1^*(\sigma)$ is characterized by:

- its action on $[(0,0,0),1]^*, [(0,0,0),2]^*, [(0,0,0),3]^*$
- the incidence matrix $\mathbf{M}_\sigma$

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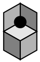

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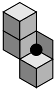

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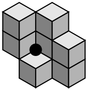

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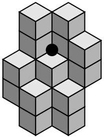

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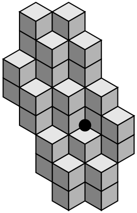

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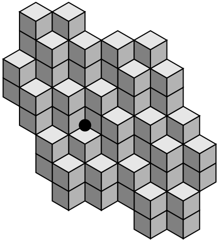

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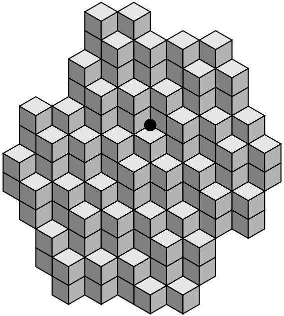

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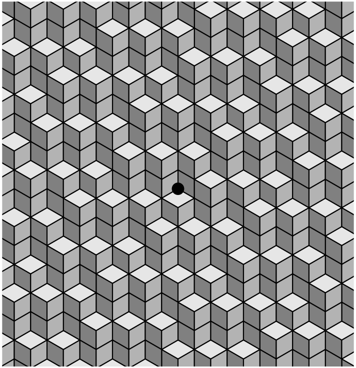
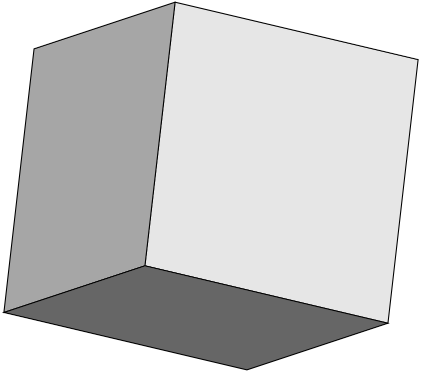


The same, with renormalization

$\pi($  $)$



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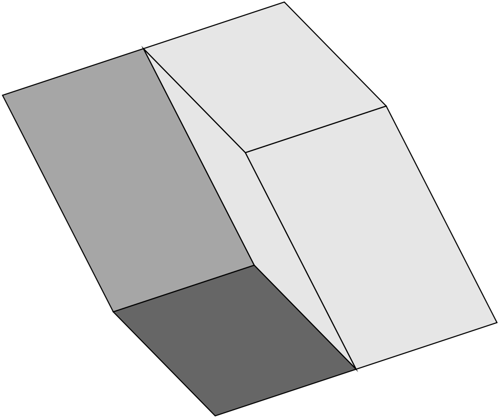
$\mathbf{M}_\sigma \pi(\mathbf{E}_1^*(\sigma)($  $))$



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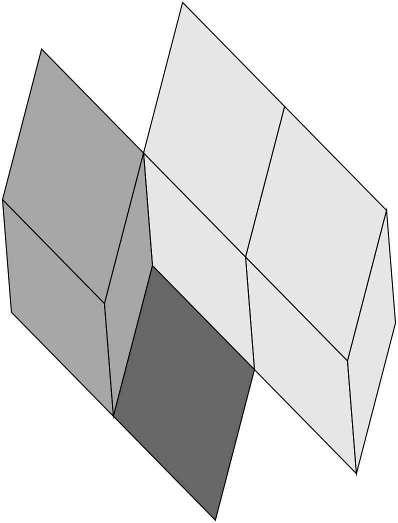
$\mathbf{M}_\sigma^2 \pi(\mathbf{E}_1^*(\sigma)^2($  $))$



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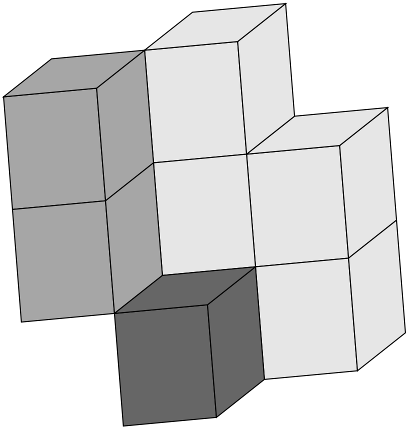

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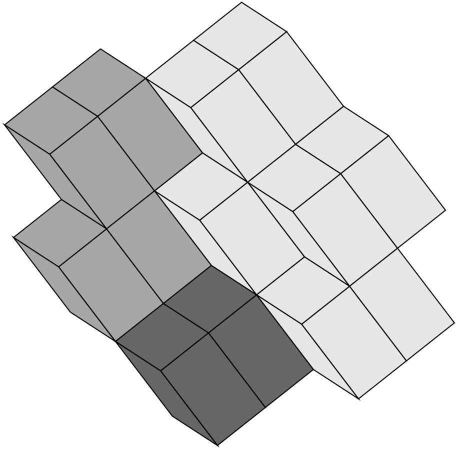

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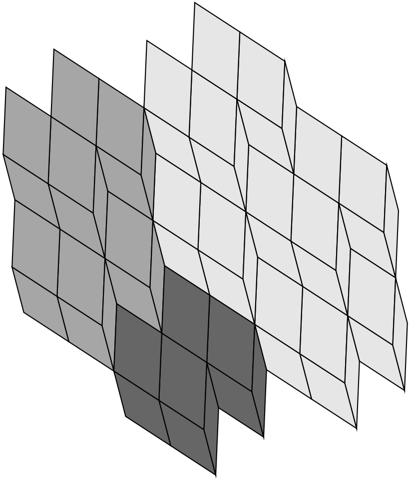

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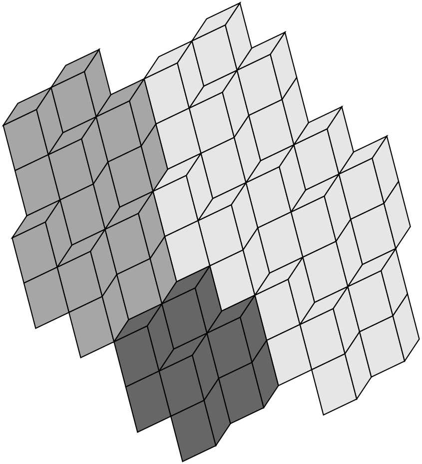

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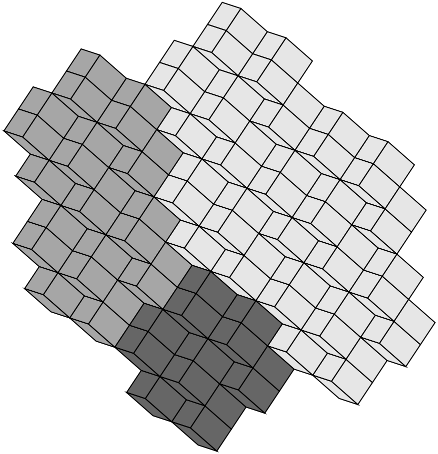
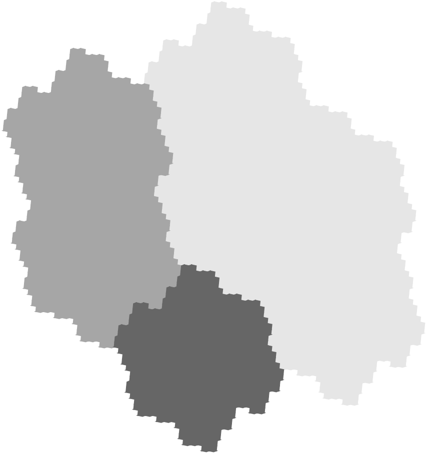


The same, with renormalization

$$\mathbf{M}_\sigma^\infty \pi(\mathbf{E}_1^*(\sigma)^\infty(\quad))$$



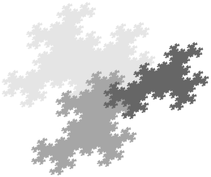
Definition of the Rauzy fractal

Let $\sigma : \{1,2,3\}^* \to \{1,2,3\}^*$ be a Pisot irreducible substitution.

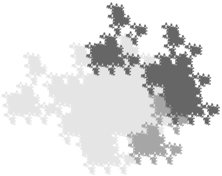
Definition [Rauzy '82, Arnoux-Ito '01]

The **Rauzy fractal** associated with σ is the set

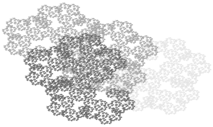
$\lim_{n\to\infty} \mathbf{M}_\sigma^n \pi(\mathbf{E}_1^*(\sigma)^n($  $))$.



$$\begin{array}{lcl} 1 & \mapsto & 12 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 1 \end{array}$$



$$\begin{array}{lcl} 1 & \mapsto & 131 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 1132 \end{array}$$



$$\begin{array}{lcl} 1 & \mapsto & 2 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 12 \end{array}$$

Position rules

We want to apply $\mathbf{E}_1^*(\sigma)$ **without computing** $\mathbf{M}_\sigma \mathbf{x}$ for every face $[\mathbf{x}, i]^*$.

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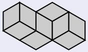
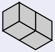
We have $\mathbf{E}_1^*(\sigma)($$) =$  and $\mathbf{E}_1^*(\sigma)($$) =$ .

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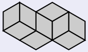
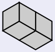
We have $\mathbf{E}_1^*(\sigma)($  $) =$  and $\mathbf{E}_1^*(\sigma)($  $) =$ .

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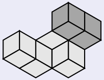
We give a **position rule**:

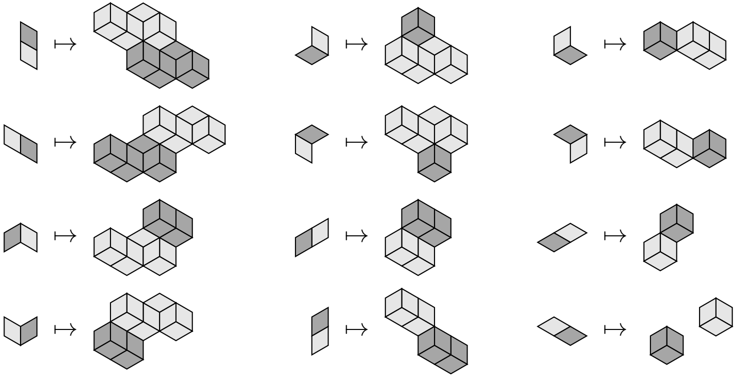
 $\mapsto$ 

This rule must agree with $\mathbf{E}_1^*(\sigma)$.

Example (continued)

Sometimes, it is possible to describe $\mathbf{E}_1^*(\sigma)$ by a **finite set of position rules** (when we only want to iterate from ).

This is the case for our example:



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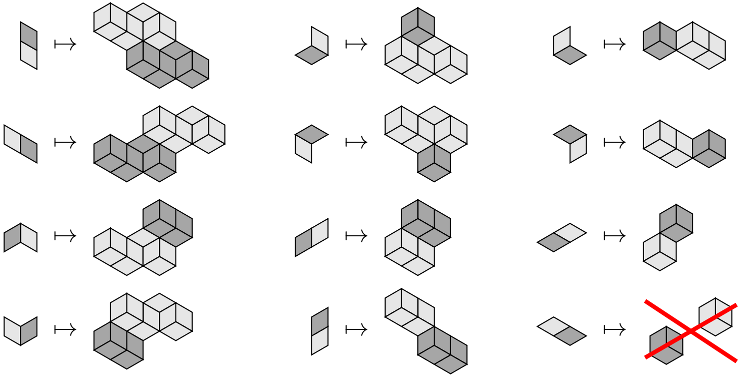

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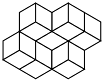

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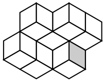
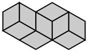
 $\mapsto$ 

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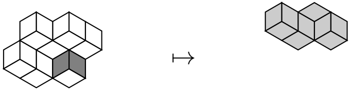

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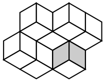

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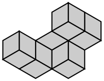

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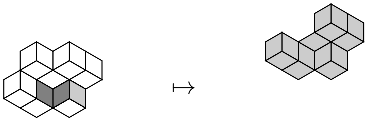

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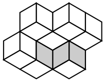

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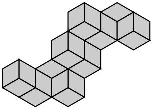

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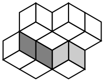

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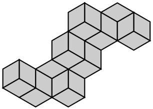

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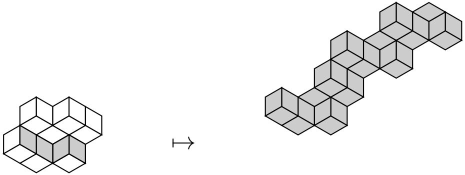

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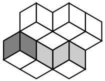

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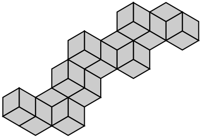

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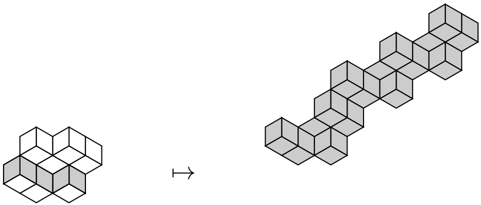

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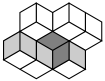
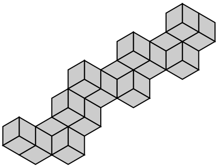
 $\mapsto$ 

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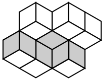
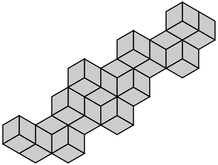


$\mapsto$



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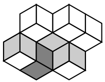

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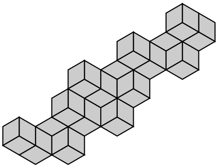

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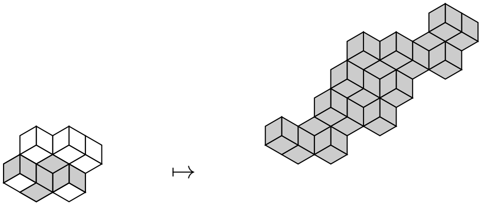

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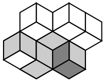

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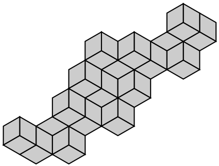

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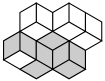

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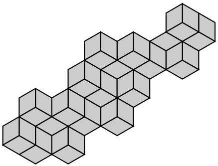

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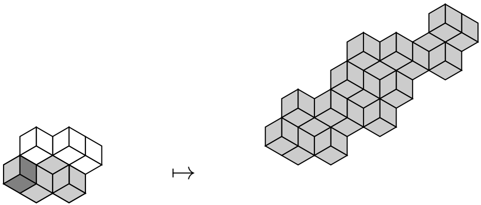

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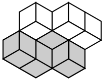

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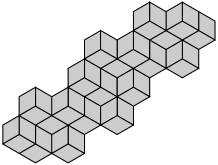

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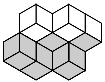
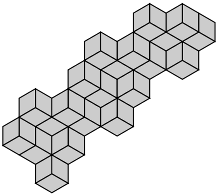
 $\mapsto$ 

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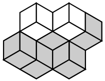

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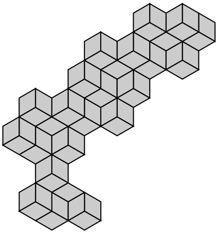

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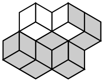

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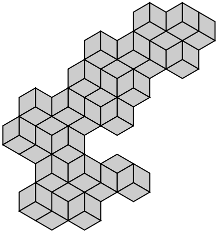

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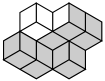

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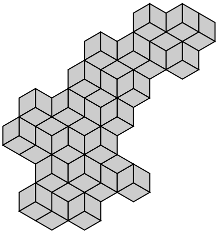

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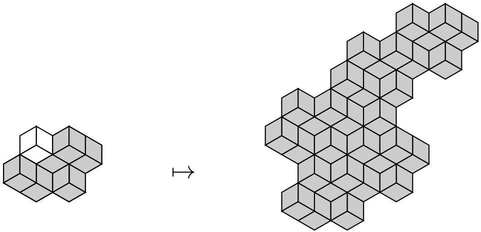

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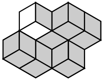

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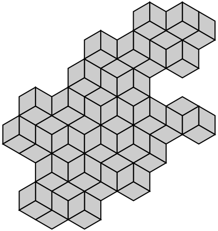

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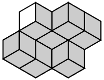

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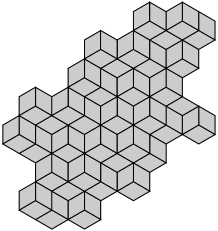

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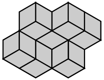

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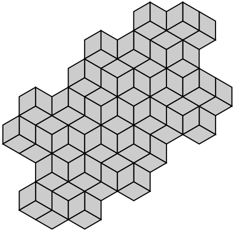

Example (continued^3)

This is an example of a **stable** set of rules (we can iterate them).

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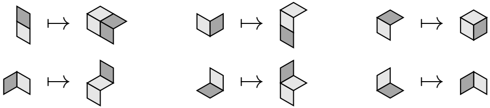
➥ The associated Rauzy fractal is connected, because connectedness is compatible with the Hausdorff limit (our sets are compact).

The same way, we can prove the connectedness of the fractal associated with:

- $\sigma : 1 \mapsto 12, 2 \mapsto 3, 1 \mapsto 1$ (minimal Pisot substitution):



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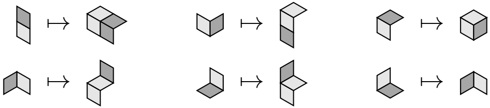
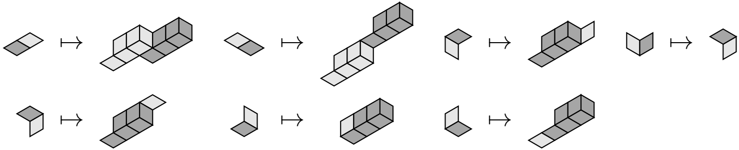


- $\sigma_{B,C} : 1 \mapsto 3, 2 \mapsto 13^2, 1 \mapsto 23^3$ (a Jacobi-Perron substitution):

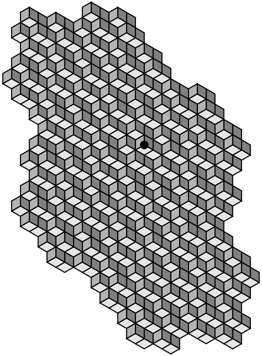


- Many other examples. . .

Let's try do deal with some families of substitutions (not only one).

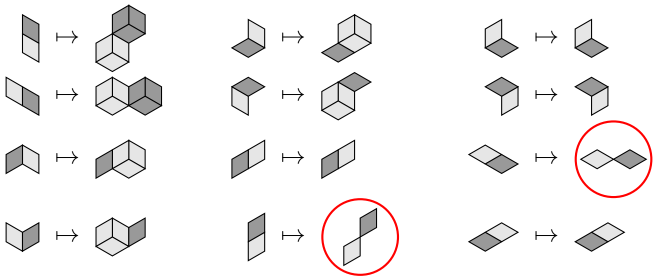
Arnoux-Rauzy substitutions

$$\mathsf{ar}_1 : \left\{ \begin{array}{lll} 1 & \mapsto & 1 \\ 2 & \mapsto & 21 \\ 3 & \mapsto & 31 \end{array} \right. \qquad \mathsf{ar}_2 : \left\{ \begin{array}{lll} 1 & \mapsto & 12 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 32 \end{array} \right. \qquad \mathsf{ar}_3 : \left\{ \begin{array}{lll} 1 & \mapsto & 13 \\ 2 & \mapsto & 23 \\ 3 & \mapsto & 3 \end{array} \right.$$



Connectedness of Arnoux-Rauzy fractals

Let's look at $\mathbf{E}_1^*(\sigma)(\mathrm{ar}_1)$:



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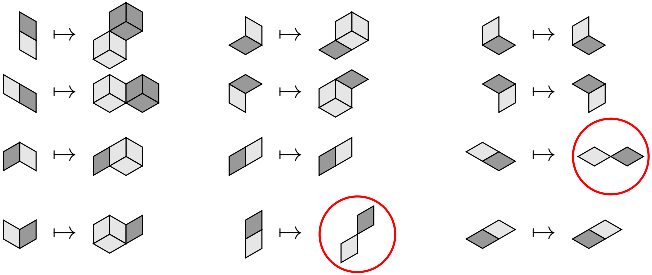
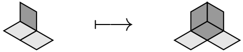


Idea: Consider larger starting patterns:



Connectedness of Arnoux-Rauzy fractals

These larger patterns yield 3 sets of 12 rules (one for each ar_i) such that:

- the 3 sets of rules for ar_1, ar_2 and ar_3 are mutually stable,
- the patterns in the rules are connected.

Connectedness of Arnoux-Rauzy fractals

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- the 3 sets of rules for ar_1, ar_2 and ar_3 are mutually stable,
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➥ We are allowed to **compose** substitutions!

Theorem (Berthé-J-Siegel)

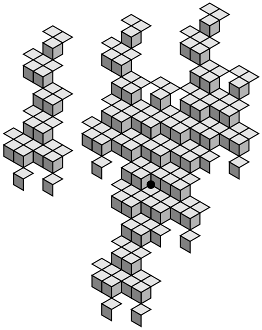
Let $\sigma = \mathsf{ar}_{i_1} \cdots \mathsf{ar}_{i_n}$ be a finite product of AR substitutions. Then:

1. $\mathbf{E}_1^*(\sigma)(\mathsf{ar}_{i_1}) \cdots \mathbf{E}_1^*(\sigma)(\mathsf{ar}_{i_n})($  $)$ is (simply) connected
2. The fractal associated with σ is connected.

Other applications

The same can be done for Jacobi-Perron substitutions ($\sigma_{B,C} : 1 \mapsto 3, 2 \mapsto 13^B, 1 \mapsto 23^C$) or the Brun substitutions [already in Ito-Ohtsuki '93, '94].

But, unfortunately, not elementary substitutions (a more general class than Arnoux-Rauzy):



Future work

- Prove the simple connectedness of Arnoux-Rauzy fractals.
- Study other continued fraction algorithm substitutions (like Brun, JP, ...).
- Prove the tiling property for some families (links with Pisot conjecture).
- **Decidability:**
  - Are the generated patches (simply) connected?
  - Is $(0, 0, 0)$ on the boundary of the patches?
  - ...
  - Same questions for fractals (many of them already answered, *cf.* [A. Siegel and J. Thuswaldner, *Topological properties of Rauzy fractals*])