Outils combinatoires pour la dynamique des systèmes S-adiques substitutifs

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FUNDIM, University of Turku, Finland

lundi 1 octobre 2012
Groupe de travail ergodique et dynamique
Orsay

Part 1:

Substitutions

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$$\sigma : \begin{cases} 1 & \mapsto & 12 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 1 \end{cases} \qquad M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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12

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In this talk: alphabet of size three $\{1, 2, 3\}$

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σ **is of Pisot type:** $\mathrm{spec}(\mathbf{M}_\sigma) = \{\beta, \beta', \beta''\}$ with:

- β is real and $\deg(\beta) = 3$
- $\beta > 1$
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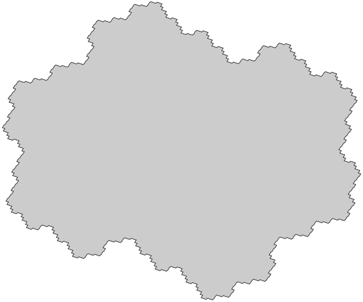
Action of $\mathbf{M}_\sigma$ on $\mathbb{R}^3$:

- A **expanding line** spanned by $\boldsymbol{v}_\beta$
- A **contractanting plane** spanned by $\boldsymbol{v}_{\beta'}, \boldsymbol{v}_{\beta''}$

Rauzy fractals

In the contracting plane of $\mathbf{M}_\sigma$ lives the **Rauzy Fractal** of σ.
[Rauzy 1982, Arnoux-Ito 2001]



- Compact
- Fractal boundary
- Self-similar structure
- Domain exchange

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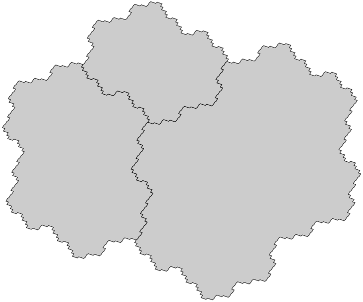

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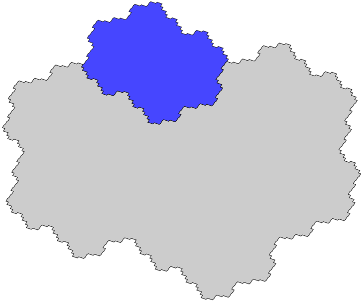

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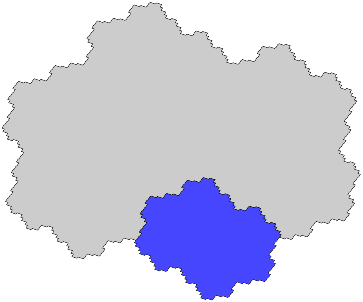

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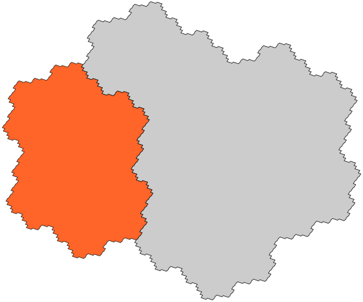

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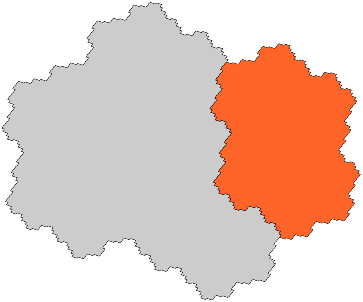

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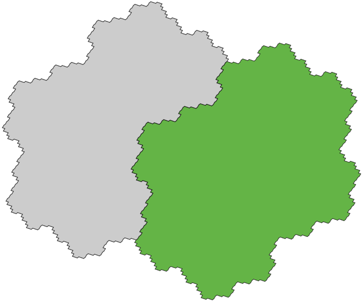

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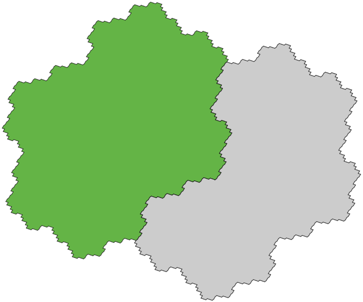

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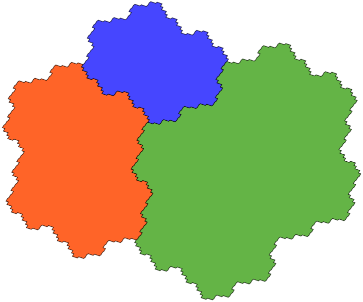

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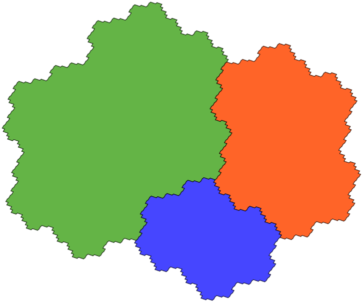

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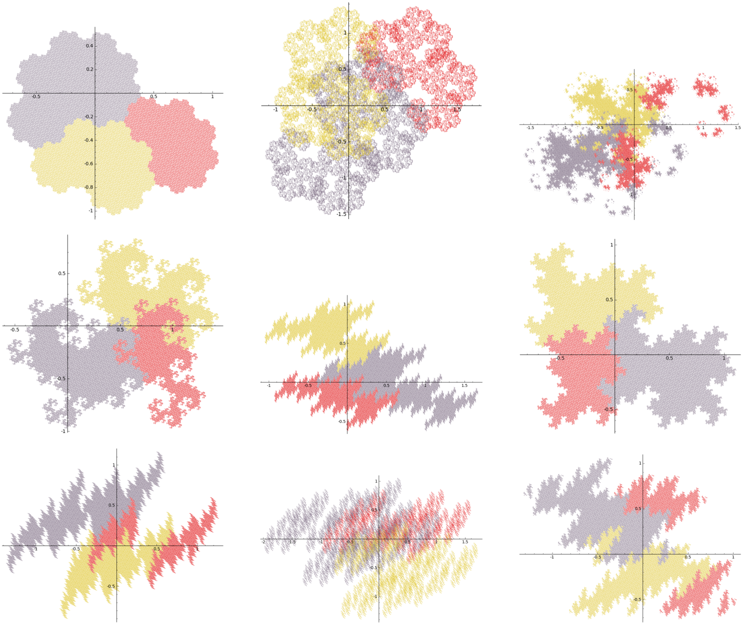

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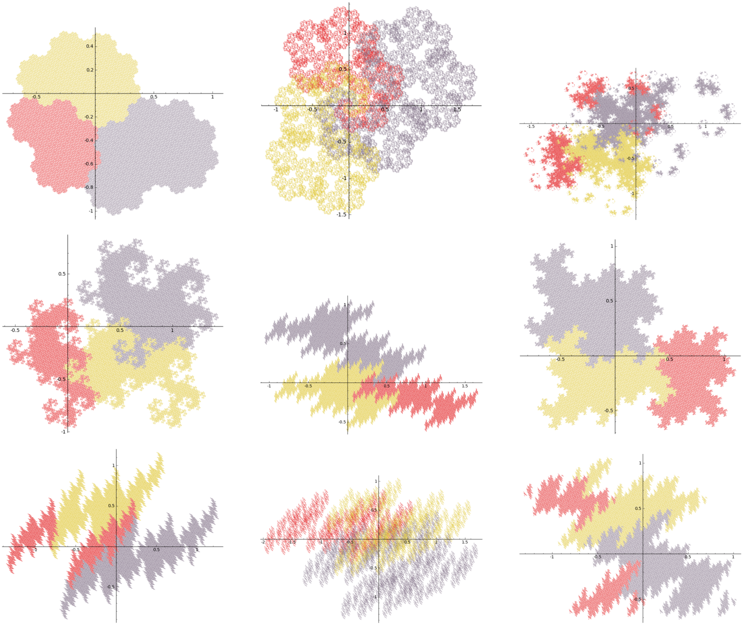

Properties, 1: dynamics of σ

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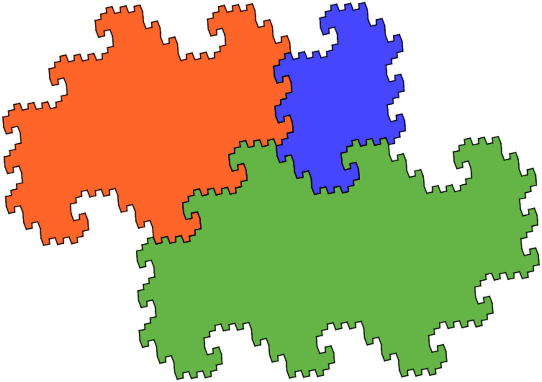
- $X_\sigma\ =\ \text{closure}(\{\text{shift}^n(x) : x \in \mathbb{Z}\})\ \subseteq\ \{1,2,3\}^{\mathbb{Z}}$

Properties, 1: dynamics of σ

- $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$
- Fixed point: $x = 121312121121213121213121212131212131212 \cdots$
- $X_\sigma = \text{closure}(\{\text{shift}^n(x) : x \in \mathbb{Z}\}) \subseteq \{1,2,3\}^{\mathbb{Z}}$
- Symbolic dynamcal system (X_σ, shift):
  - minimal sytem,
  - zero entropy,
  - no periodic points...

Properties, 1: dynamics of σ



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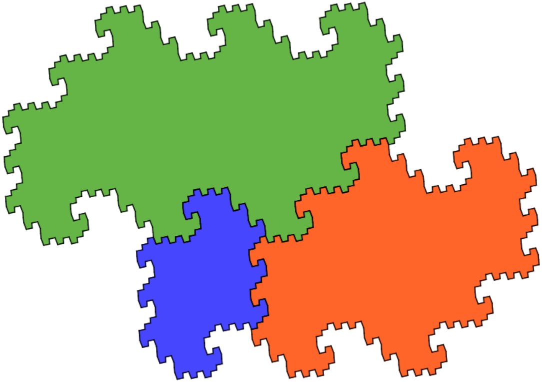

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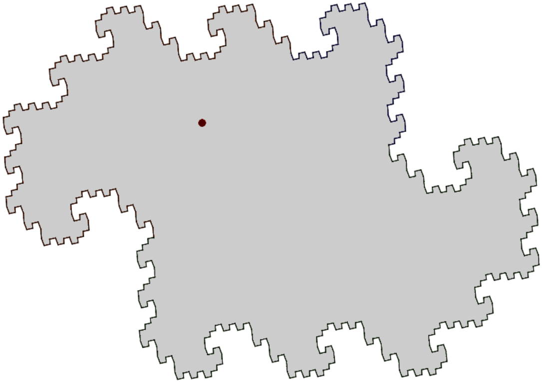

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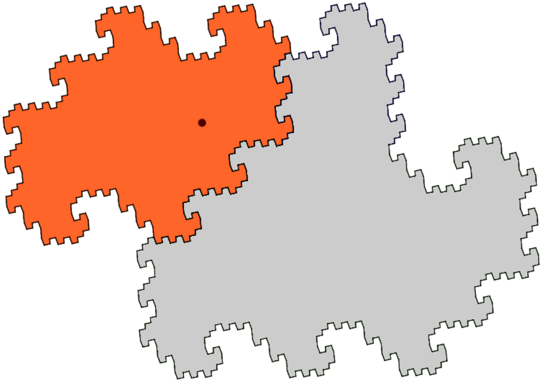

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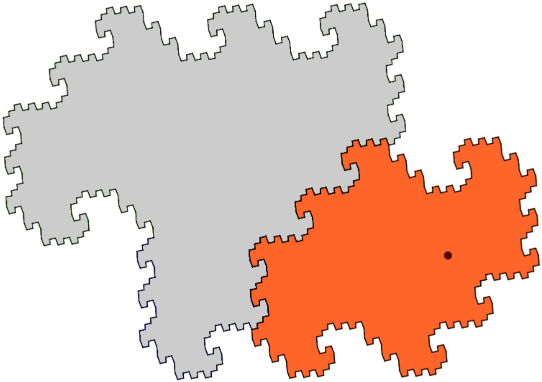

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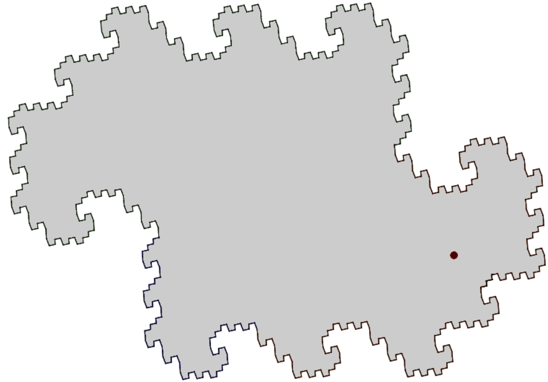
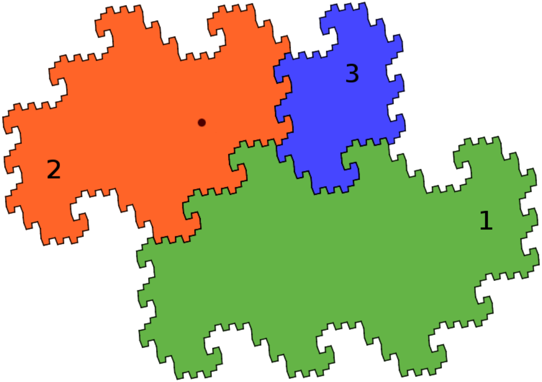


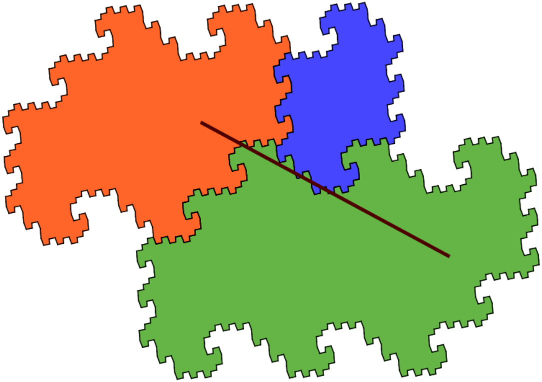
Properties, 1: dynamics of σ

Orbit: 2



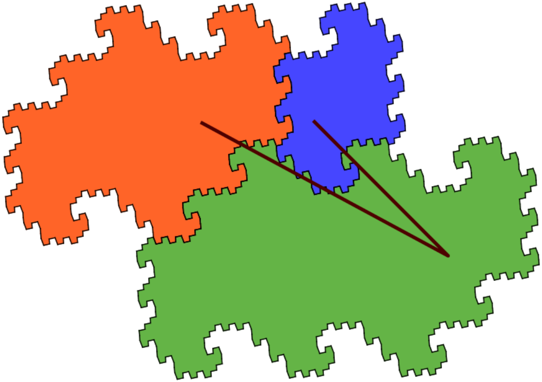
Properties, 1: dynamics of σ

Orbit: 21



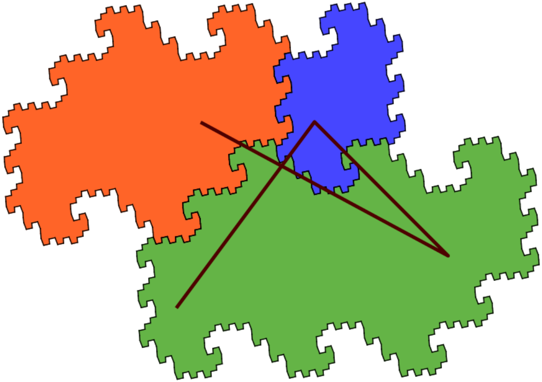
Properties, 1: dynamics of σ

Orbit: 213



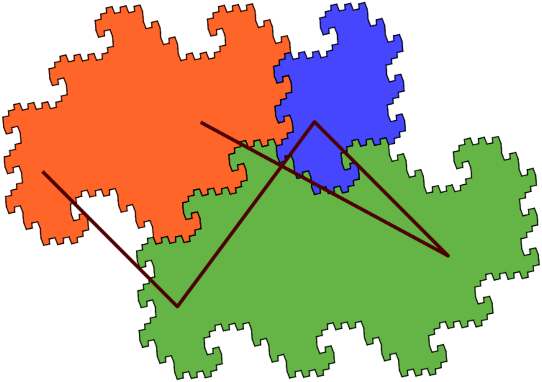
Properties, 1: dynamics of σ

Orbit: 2131



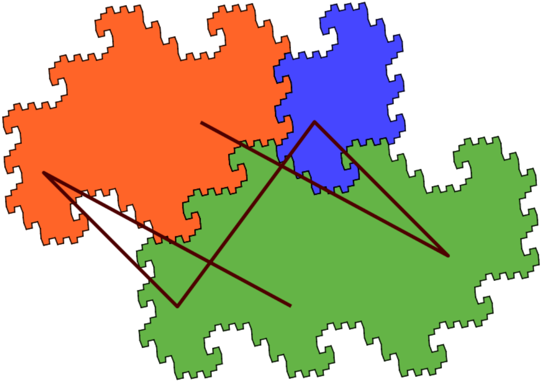
Properties, 1: dynamics of σ

Orbit: 21312



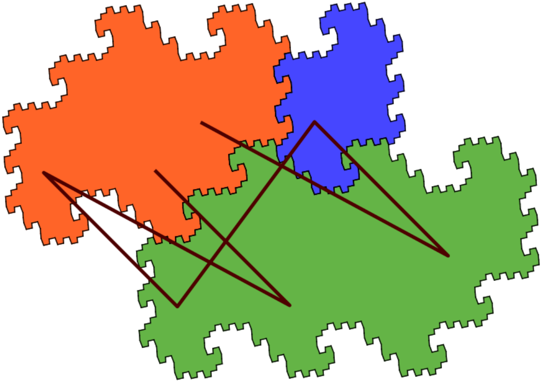
Properties, 1: dynamics of σ

Orbit: 213121



Properties, 1: dynamics of σ

Orbit: 2131212



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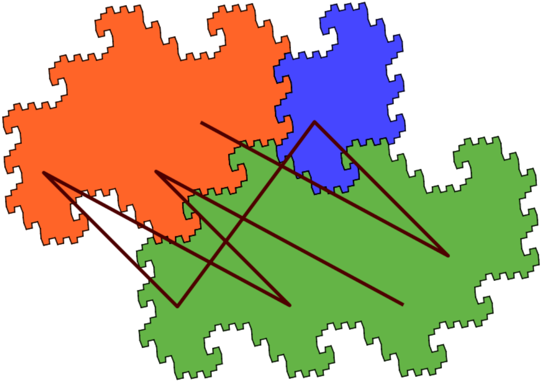
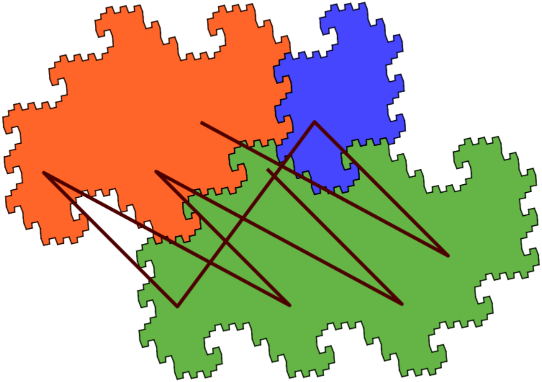
Orbit: 21312121



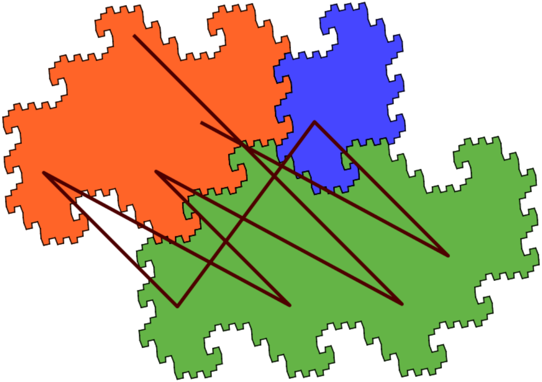
Properties, 1: dynamics of σ

Orbit: 213121211



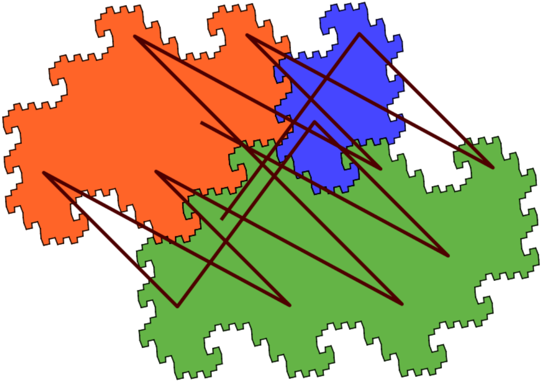
Properties, 1: dynamics of σ

Orbit: 2131212112



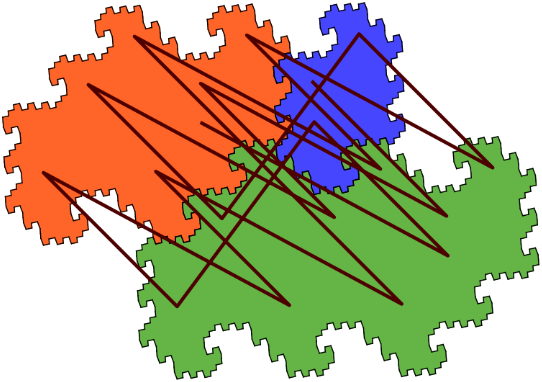
Properties, 1: dynamics of σ

Orbit: 213121211212131



Properties, 1: dynamics of σ

Orbit: 2131212112121312121213



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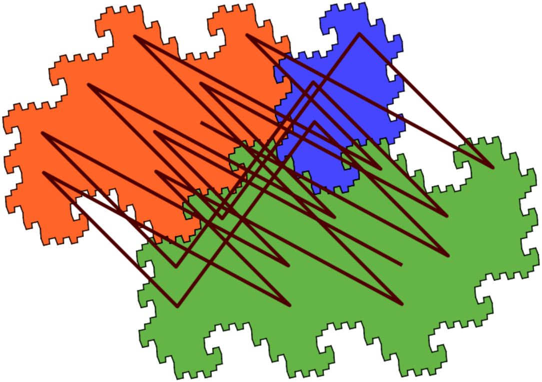

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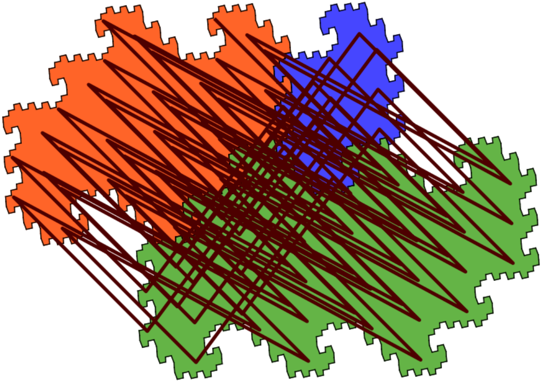
Orbit: $\cdots 21312121121213121213121 21\cdots \in X_\sigma \subseteq \{1,2,3\}^{\mathbb{Z}}$



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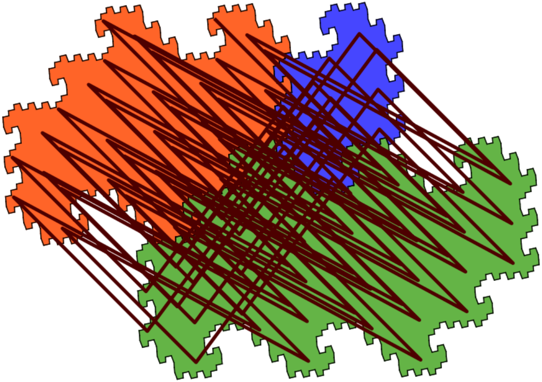
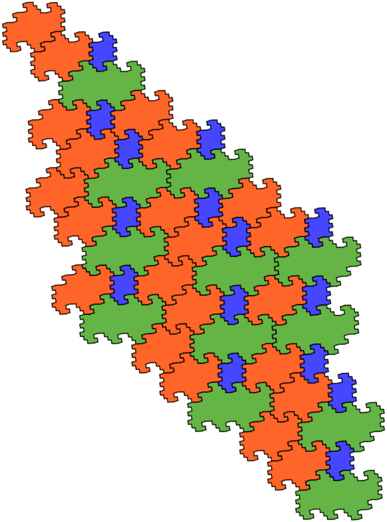


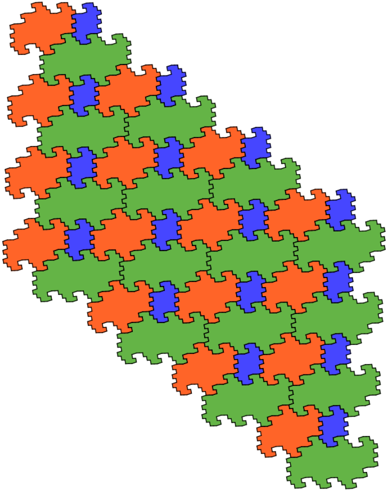
$\left(X_\sigma\,,\ \text{shift}\right) \quad \cong \quad \Big($  $,\ \text{exchange}\Big)$

Tilings

Self-similar tiling (aperiodic):

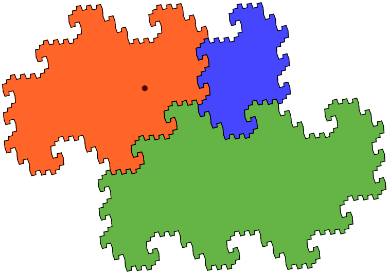


Periodic tiling:

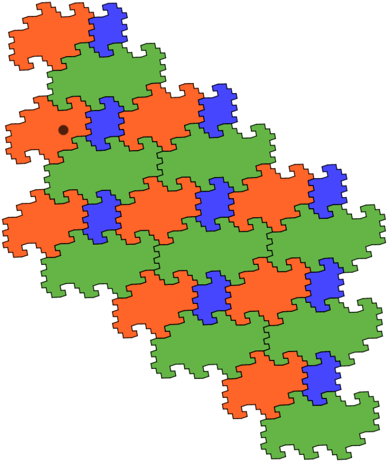


Properties, 1: dynamics of σ

(1) Domain exchange:



(2) Translation on the torus $\mathbb{T}^2$:



(3) Shift:

$$\cdots \underline{2}131212112 \cdots \ \in X_\sigma$$

Properties, 1: dynamics of σ

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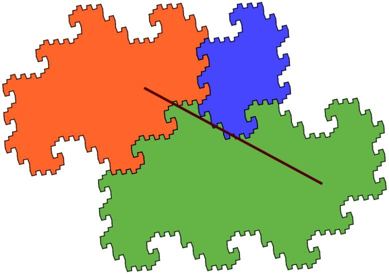

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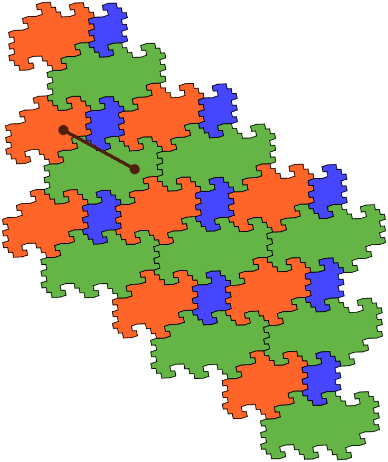

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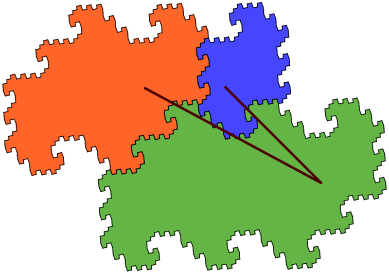

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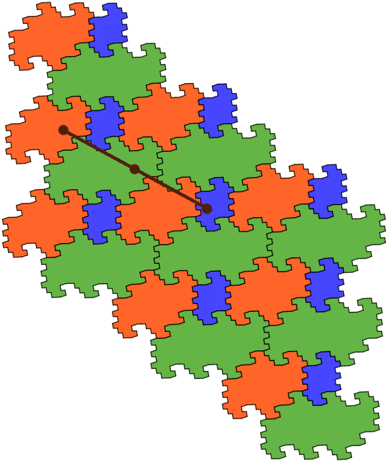

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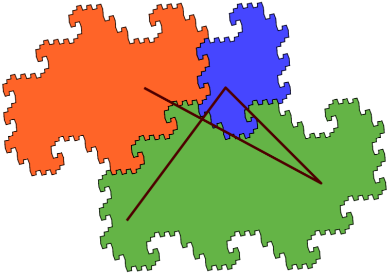

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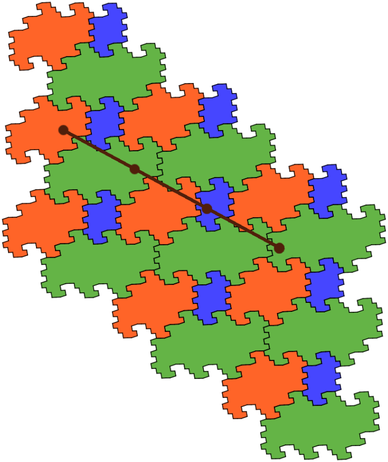

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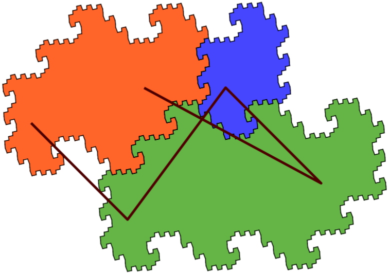

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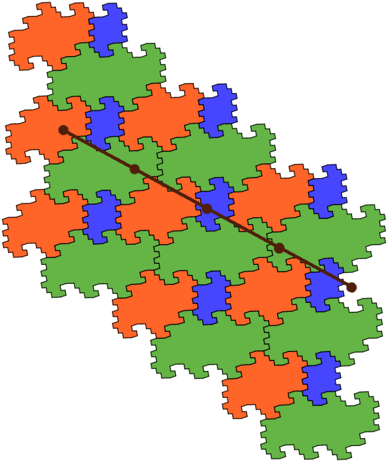

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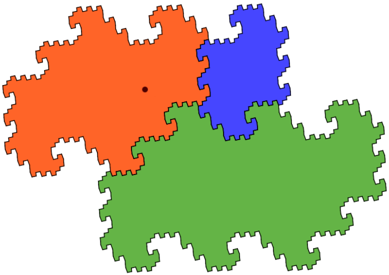

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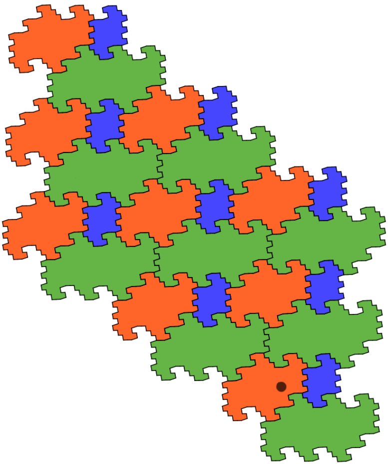

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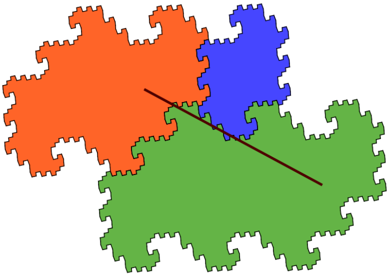

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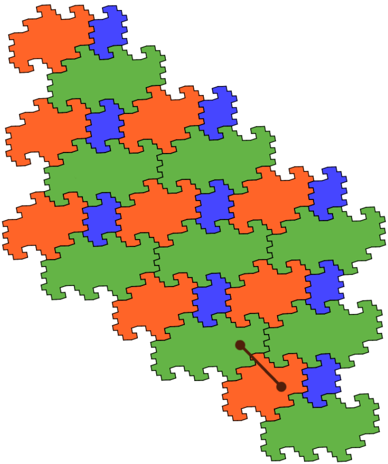

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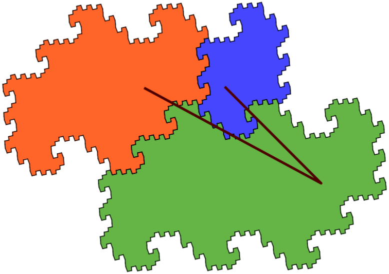

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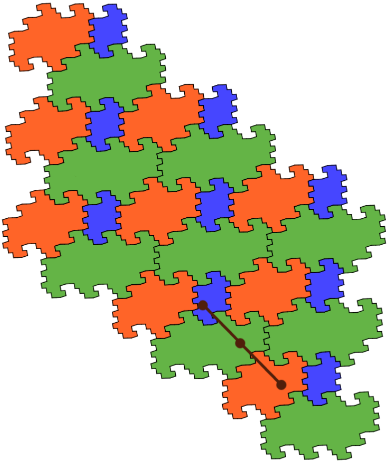

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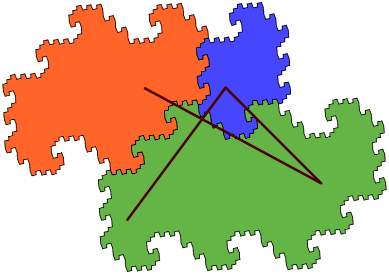

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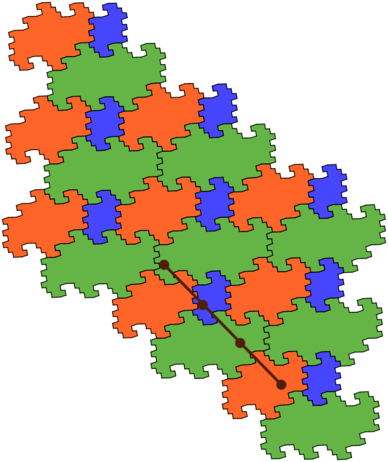

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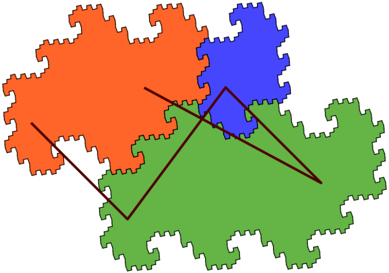

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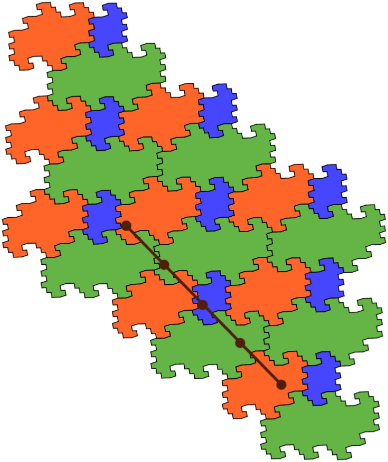

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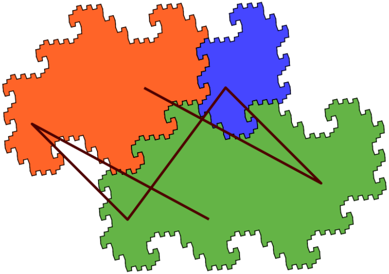

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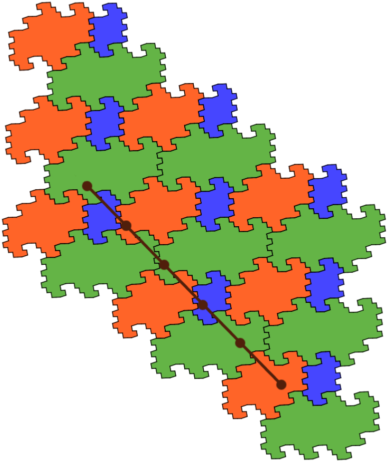

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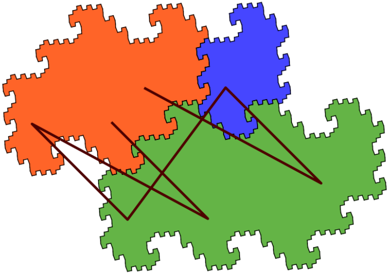

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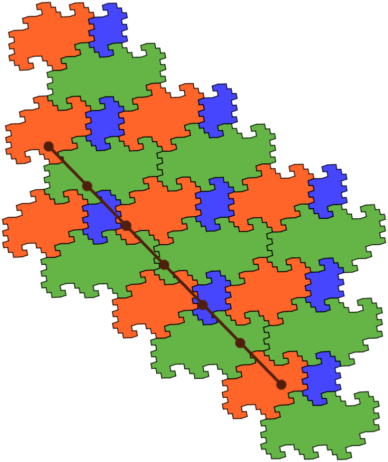

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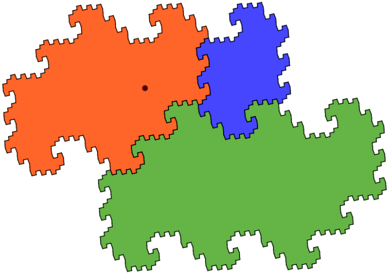

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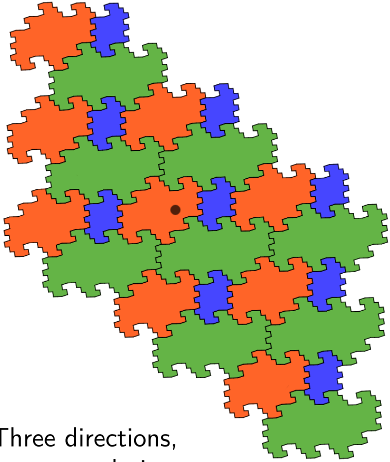

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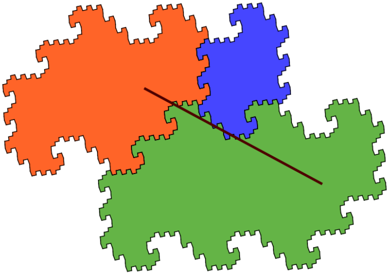

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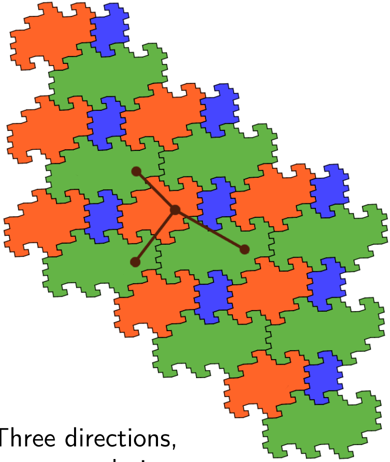

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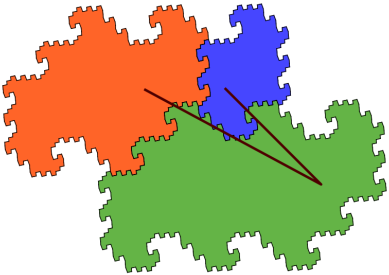

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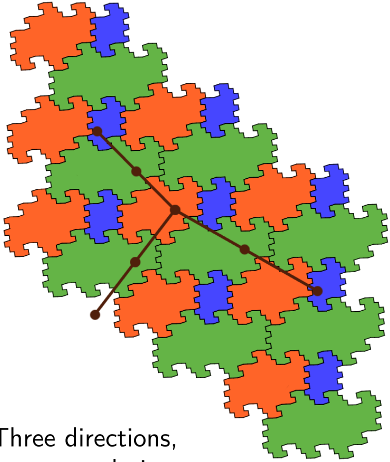

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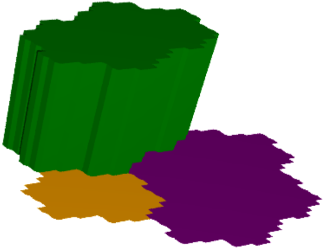

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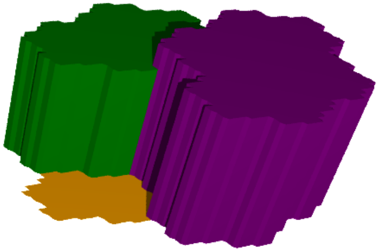

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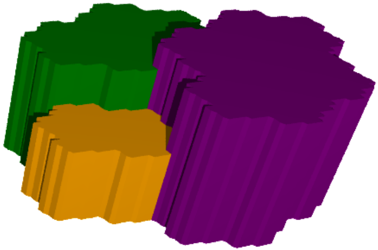

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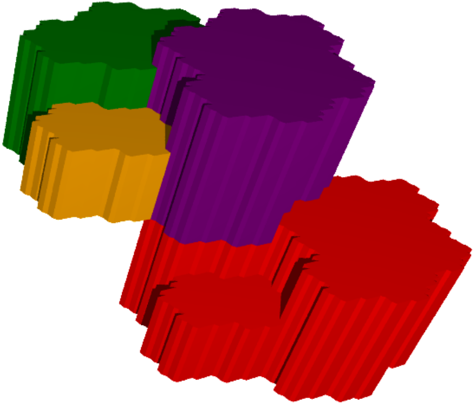

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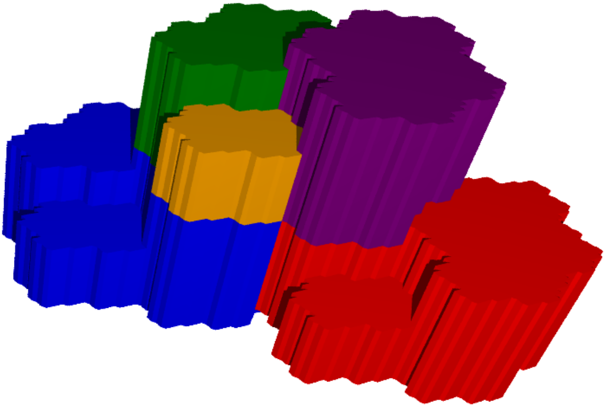

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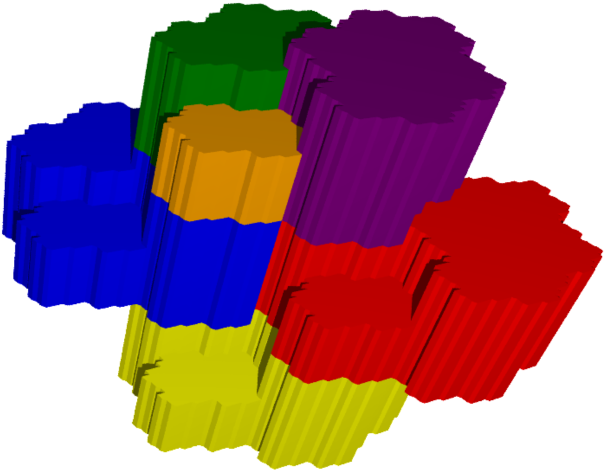

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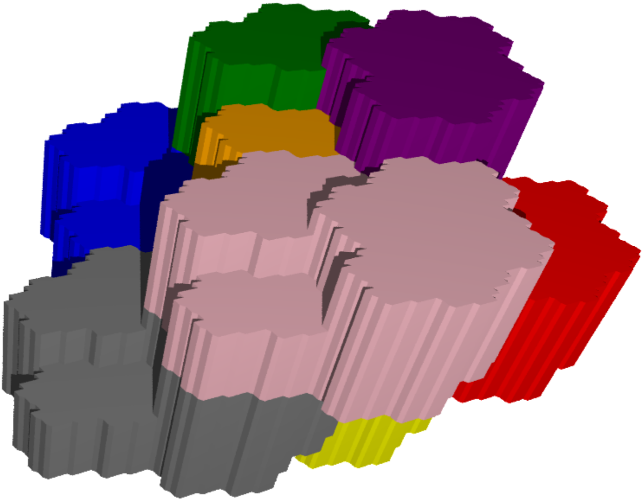

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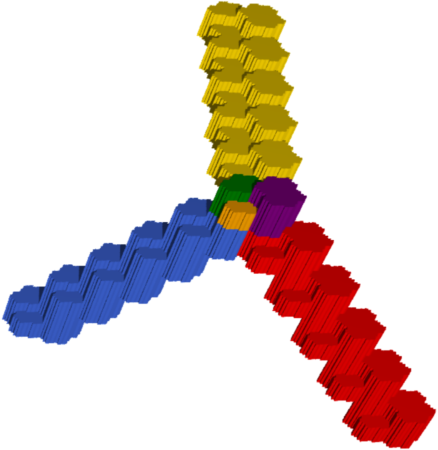

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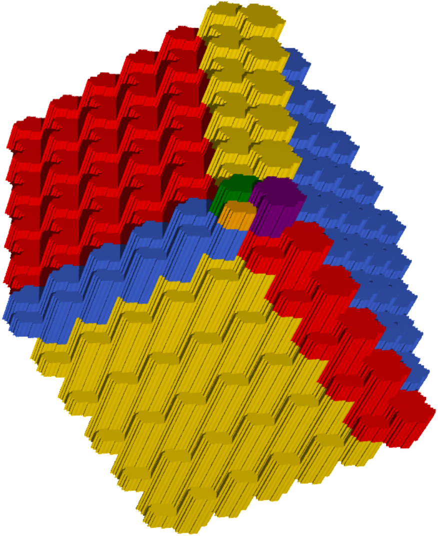

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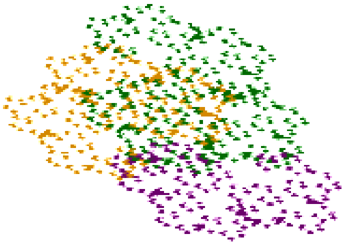

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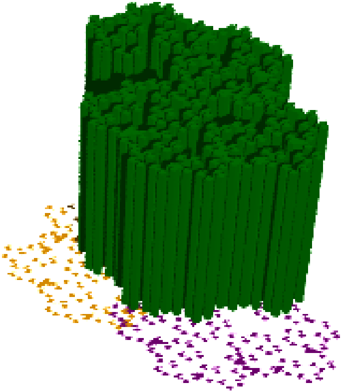

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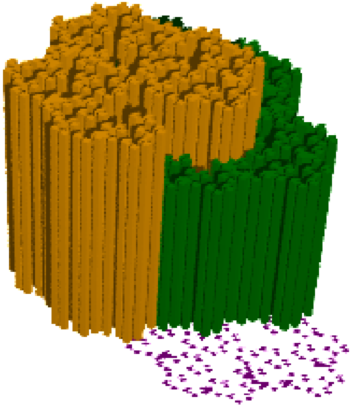

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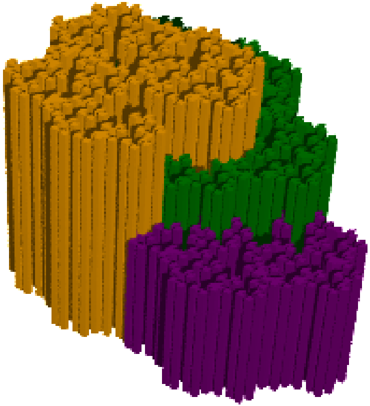

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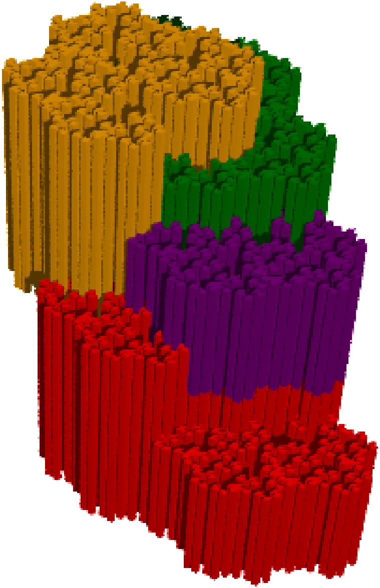

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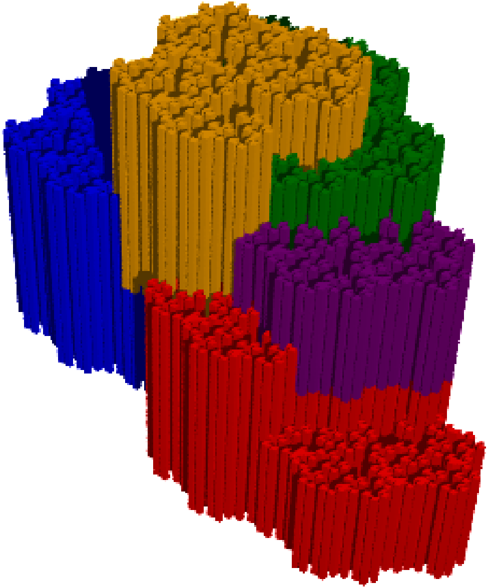

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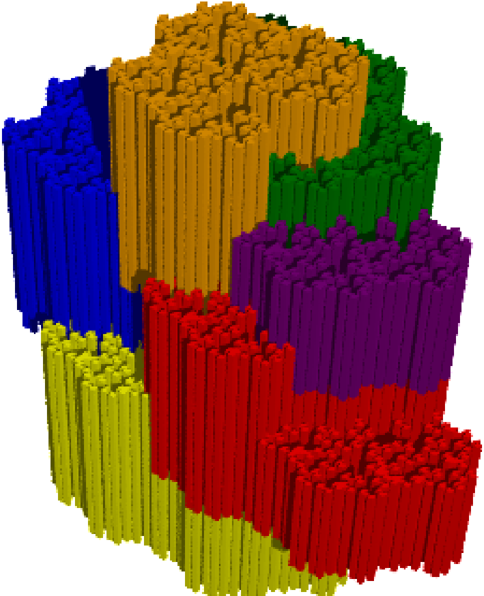

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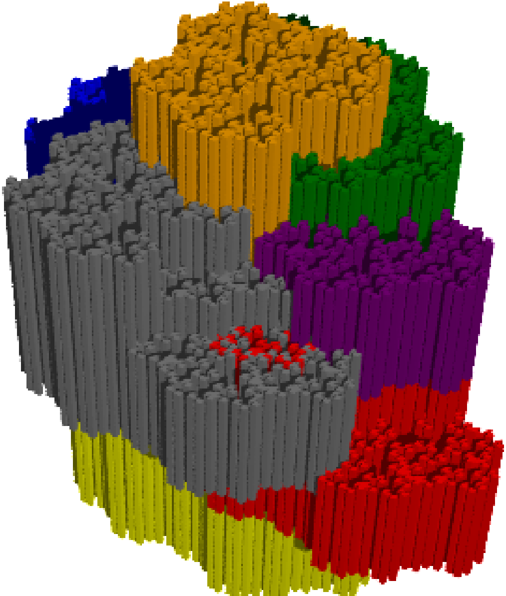

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Question

- All this works for a single given σ.
- **How to deal with infinite families** such as

$$\{\sigma_{i_1} \cdots \sigma_{i_n} : i_k \in \{1, 2, 3\}\}$$

  (Finite products from a given set of substitutions.)
- Examples:
  - Arnoux-Rauzy, $\sigma_i : \begin{cases} j \mapsto j & \text{if } j = i \\ j \mapsto ji & \text{if } j \neq i \end{cases} \quad (i = 1, 2, 3)$
  - Brun, $\begin{array}{l} \sigma_1 : 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 23 \\ \sigma_2 : 1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 23 \\ \sigma_3 : 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 13 \end{array}$
  - Jacobi-Perron, $\sigma_{B,C} : 1 \mapsto 3, 2 \mapsto 13^B, 3 \mapsto 23^C \quad (0 \leqslant B \leqslant C,\ C \neq 0)$
  - ...

Part 2:

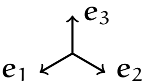
Tools:

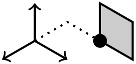
Dual substitutions

Unit faces

A **unit face** $[x, i]^*$ consists of:

- a **position** $x \in \mathbb{Z}^3$;
- a **type** $i \in \{1, 2, 3\}$.





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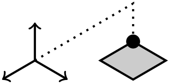

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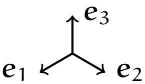

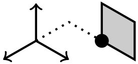

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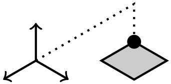

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$[x, 1]^* \;=\; \{x + \lambda e_2 + \mu e_3 : \lambda, \mu \in [0, 1]\} \;=$ 

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Main tool: Dual substitutions $E_1^*(\sigma)$

Definition [Arnoux-Ito 2001]

Let $\sigma : \{1,2,3\}^* \to \{1,2,3\}^*$ such that $\det(\mathbf{M}_\sigma) = \pm 1$.

$$E_1^*(\sigma)([\mathbf{x}, i]^\star) = \bigcup_{(p,j,s)\in\mathcal{A}^\star\times\mathcal{A}\times\mathcal{A}^\star \;:\; \sigma(j)=pis} [\mathbf{M}_\sigma^{-1}(\mathbf{x} + \ell(s)), j]^\star,$$

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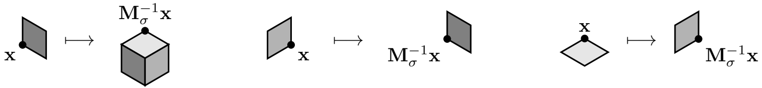
where $\ell : \{1,2,3\}^* \to \mathbb{Z}_+^3$, $w \mapsto (|w|_1, |w|_2, |w|_3)$.

Example: $E_1^*(\sigma)$ for $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$

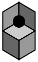
$$\begin{array}{lcl} [\mathbf{x},1]^* & \mapsto & \mathbf{M}_\sigma^{-1}\mathbf{x} + [(1,0,-1),1]^* \cup [(0,1,-1),2]^* \cup [(0,0,0),3]^* \\ [\mathbf{x},2]^* & \mapsto & \mathbf{M}_\sigma^{-1}\mathbf{x} + [(0,0,0),1]^* \\ [\mathbf{x},3]^* & \mapsto & \mathbf{M}_\sigma^{-1}\mathbf{x} + [(0,0,0),2]^* \end{array}$$



$E_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

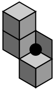
$E_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

$E_1^*(\sigma)($$)$



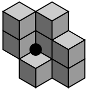
$E_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

$E_1^*(\sigma)^2($  $)$



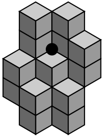
$E_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

$E_1^*(\sigma)^3($  $)$



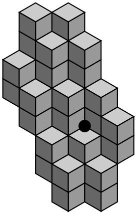
$E_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

$E_1^*(\sigma)^4($  $)$



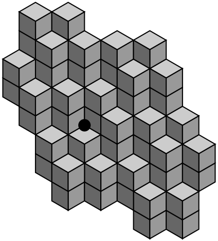
$E_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

$E_1^*(\sigma)^5($$)$



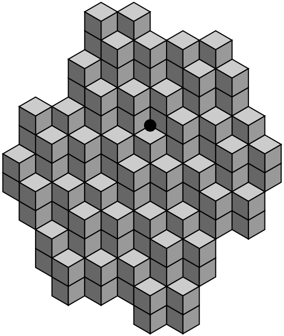
$E_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

$E_1^*(\sigma)^6($  $)$



$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

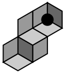
$\mathbf{E}_1^*(\sigma)^7($  $)$



$$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112)$$

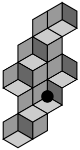
$E_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112)$

$E_1^*(\sigma)($$)$



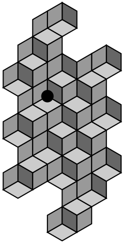
$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112)$

$\mathbf{E}_1^*(\sigma)^2($  $)$



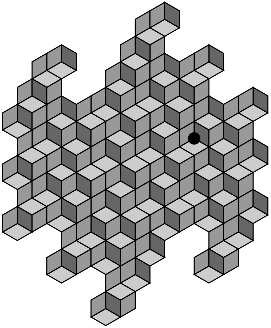
$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112)$

$\mathbf{E}_1^*(\sigma)^3($$)$



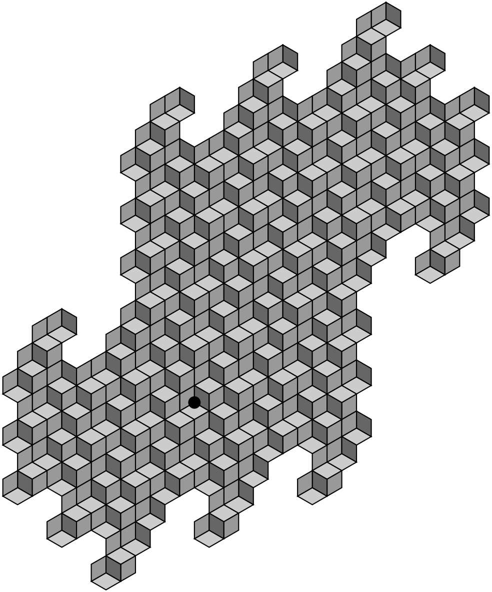
$E_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112)$

$E_1^*(\sigma)^4($$)$



$E_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112)$

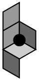
$E_1^*(\sigma)^5($$)$



$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1)$

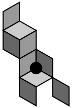
$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)($  $)$



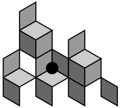
$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)^2($  $)$



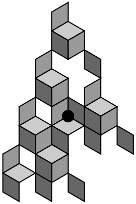
$E_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1)$

$E_1^*(\sigma)^3($$)$



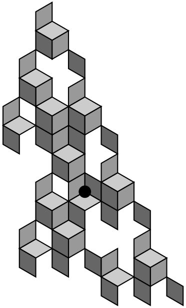
$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)^4($  $)$



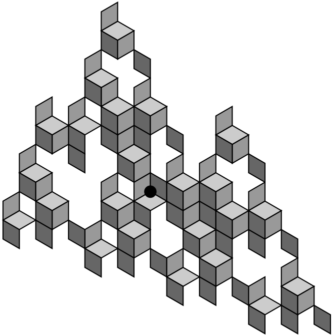
$E_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1)$

$E_1^*(\sigma)^5($$)$



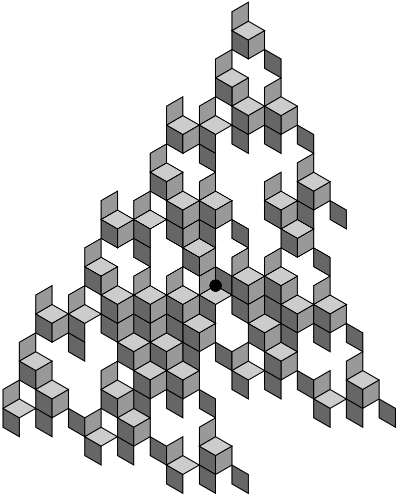
$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)^6($  $)$



$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)^7($  $)$



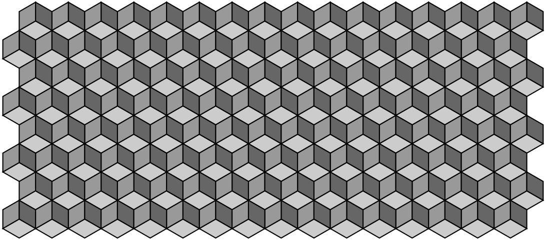
Discrete planes

Let $\boldsymbol{v} \in \mathbb{R}^3_{>0}$. The **discrete plane** $\Gamma_{\boldsymbol{v}}$ of normal vector $\boldsymbol{v}$ is

the discrete surface that "intersects" the plane $\mathcal{P}_{\boldsymbol{v}}$.

In other words: $\Gamma_{\boldsymbol{v}} = \{[\boldsymbol{x}, i]^* : 0 \leqslant \langle \boldsymbol{x}, \boldsymbol{v} \rangle < \langle \boldsymbol{e}_i, \boldsymbol{v} \rangle\}$.

$\Gamma_{(1,1,1)}$



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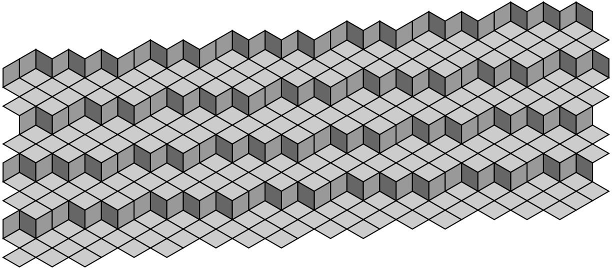
$\Gamma_{(1,\sqrt{2},\sqrt{17})}$



$E_1^*(\sigma)$ and discrete planes

Theorem [Arnoux-Ito 2001, Fernique 2007]

$E_1^*(\sigma)^n($  $) \subseteq$ **a discrete plane** for all $n \geqslant 0$.

$E_1^*(\sigma)$ and discrete planes

Theorem [Arnoux-Ito 2001, Fernique 2007]

$E_1^*(\sigma)^n($$) \subseteq$ **a discrete plane** for all $n \geqslant 0$.

Stronger: $E_1^*(\sigma)(\Gamma_{\boldsymbol{v}}) = \Gamma_{{}^t\boldsymbol{M}_\sigma \boldsymbol{v}}$.

$E_1^*(\sigma)$ and discrete planes

Theorem [Arnoux-Ito 2001, Fernique 2007]

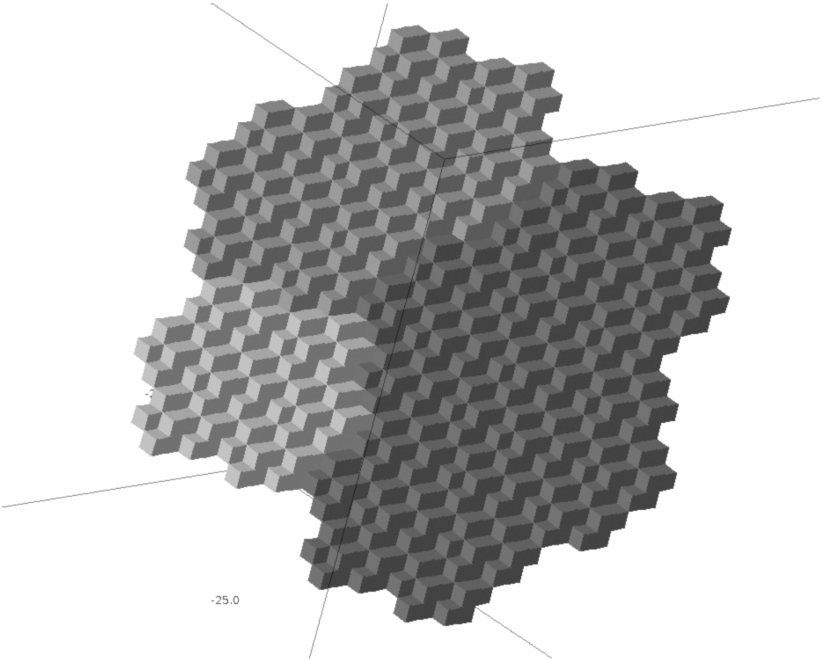
$E_1^*(\sigma)^n($$)$ $\subseteq$ **a discrete plane** for all $n \geqslant 0$.

Stronger: $E_1^*(\sigma)(\Gamma_{\mathbf{v}}) = \Gamma_{{}^t\mathbf{M}_\sigma \mathbf{v}}$.

Theorem [Arnoux-Ito 2001]

$[\mathbf{x}, i]^* \neq [\mathbf{y}, j]^* \in \Gamma_{\mathbf{v}}$

$\Longrightarrow$ $E_1^*(\sigma)([\mathbf{x}, i]^*)$ and $E_1^*(\sigma)([\mathbf{y}, j]^*)$ are disjoint.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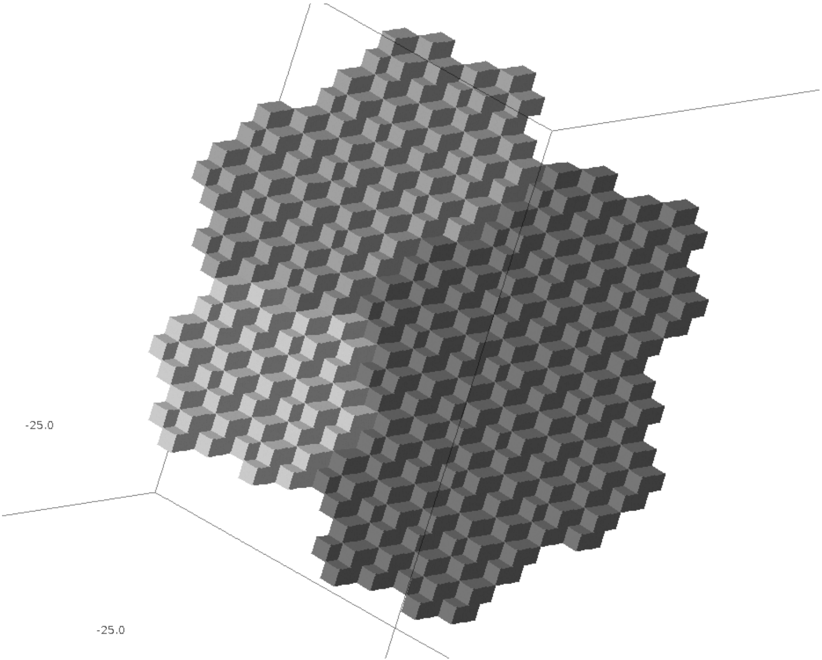
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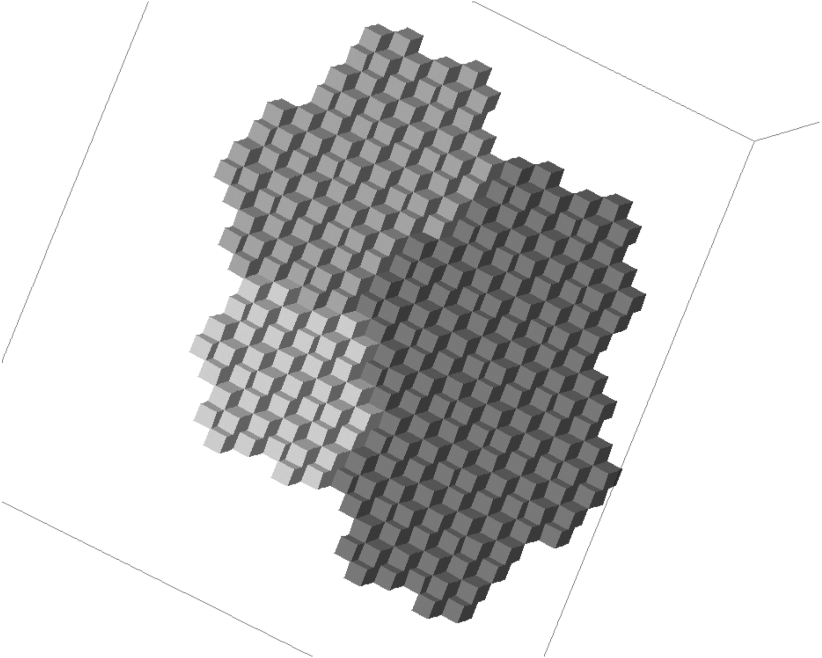

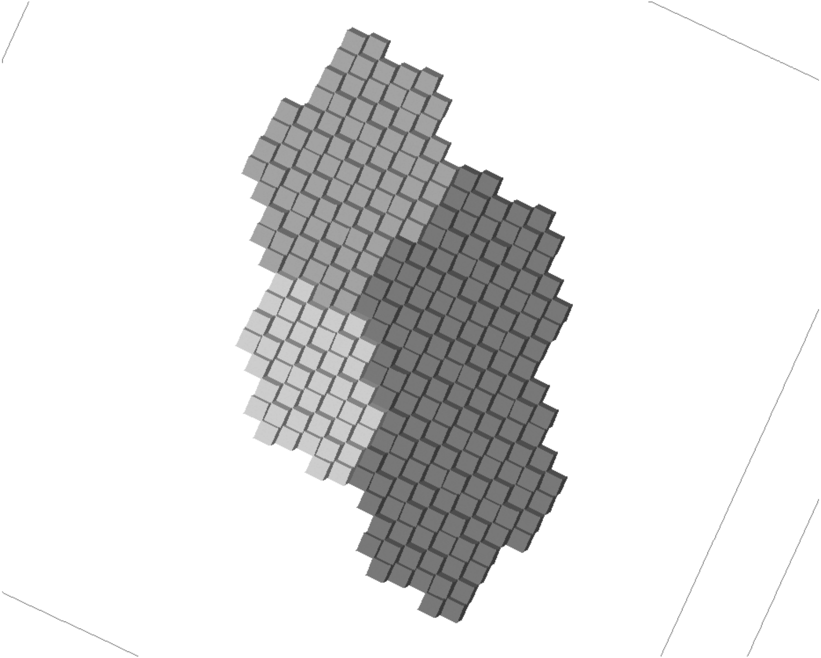

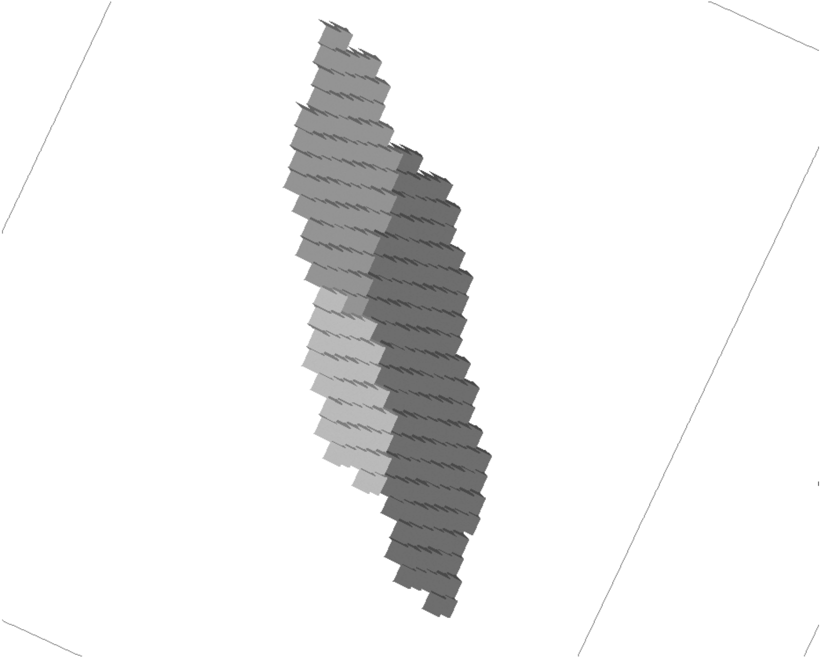

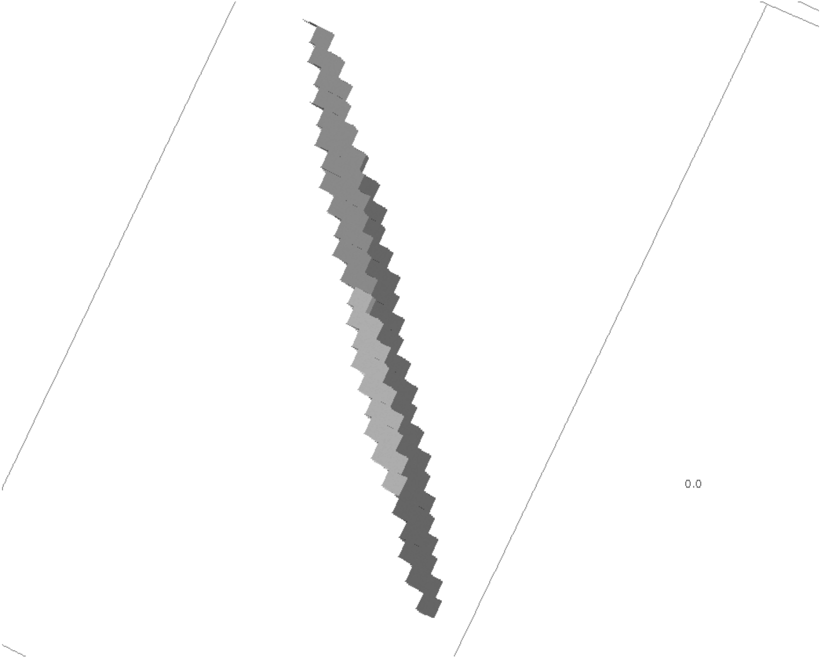

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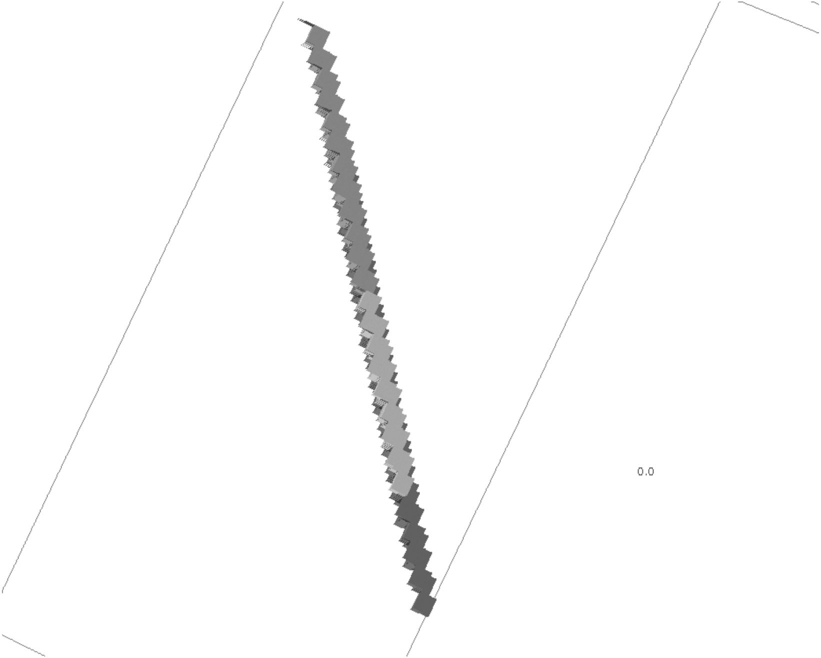
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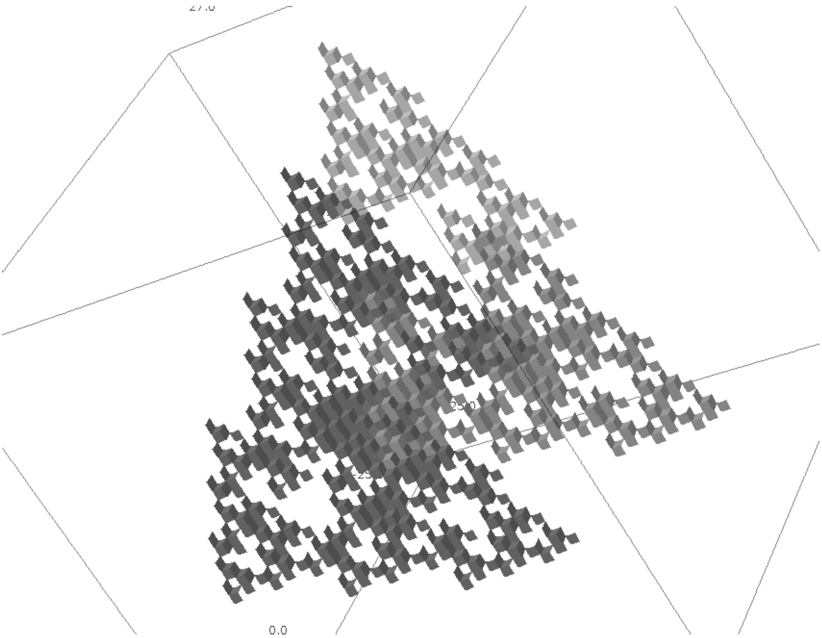
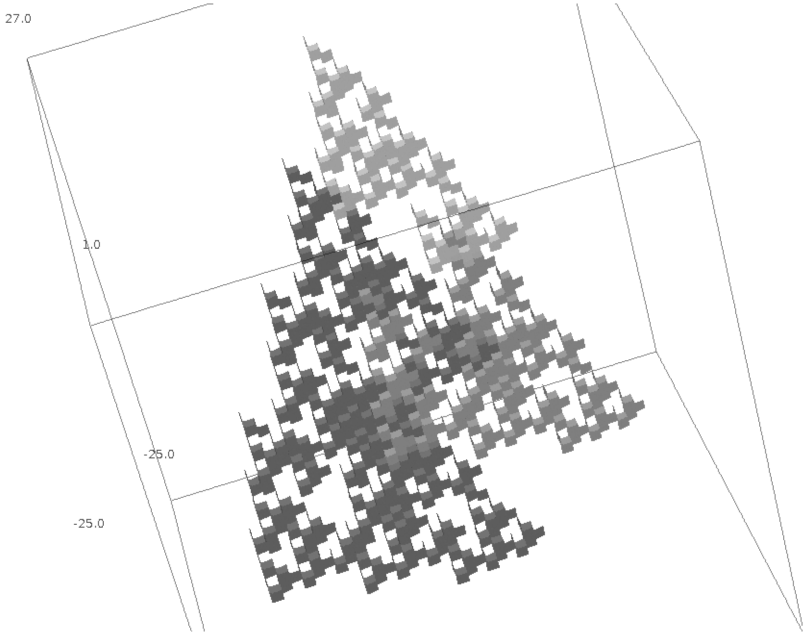


27.0
0.0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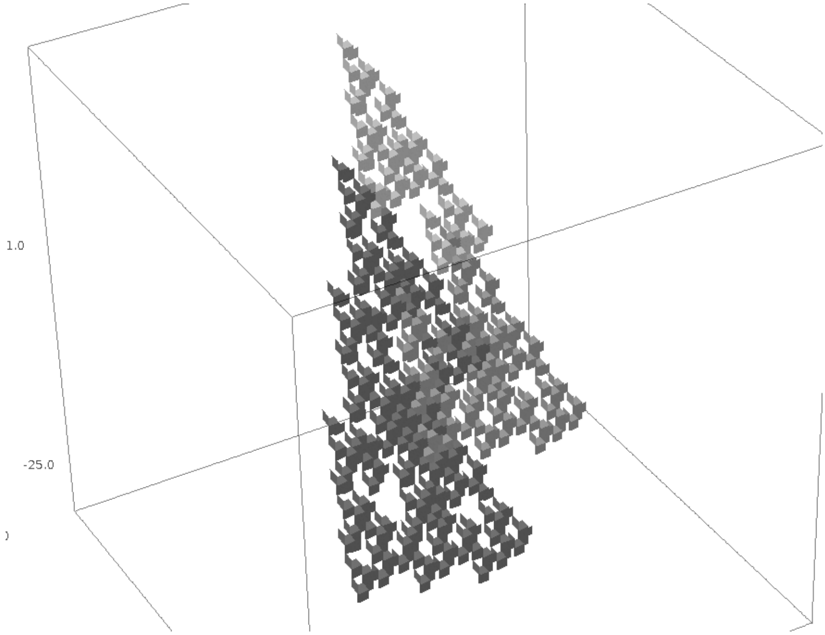
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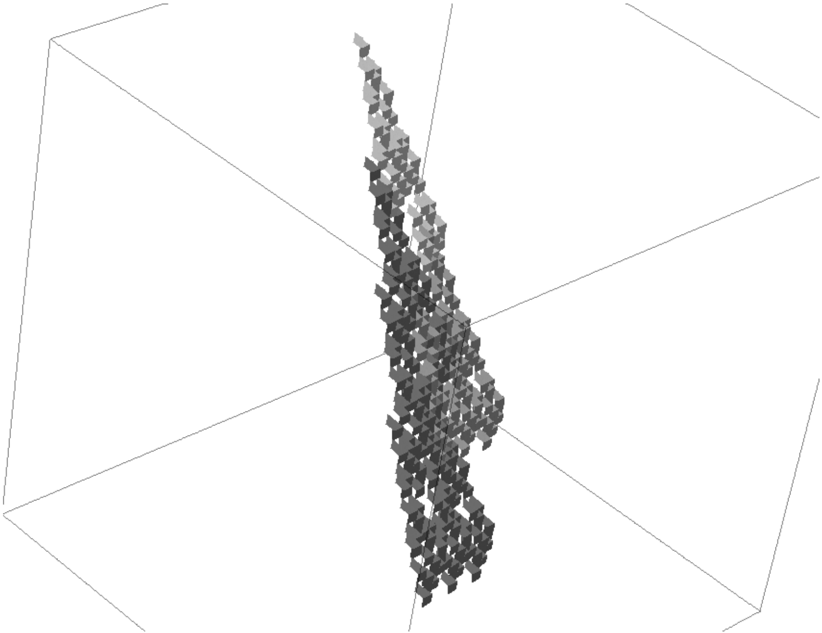
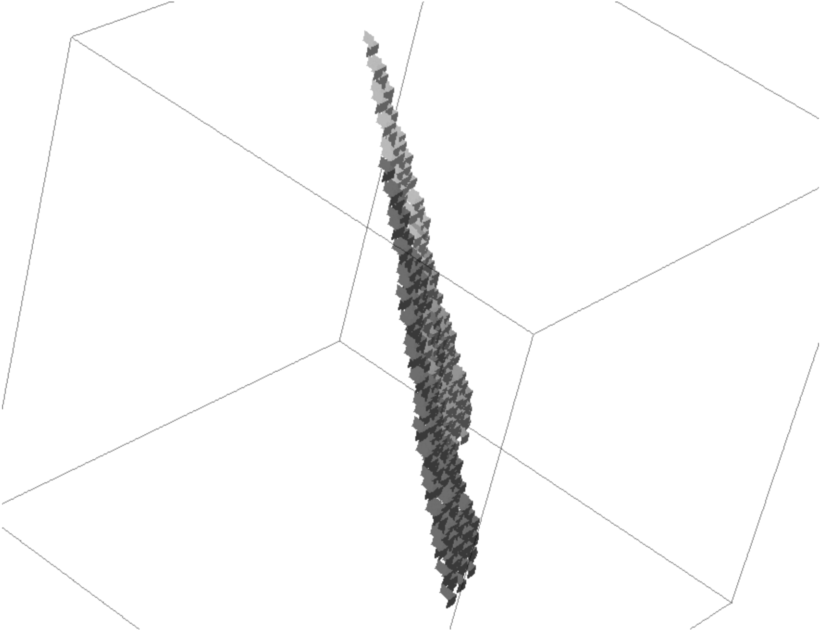
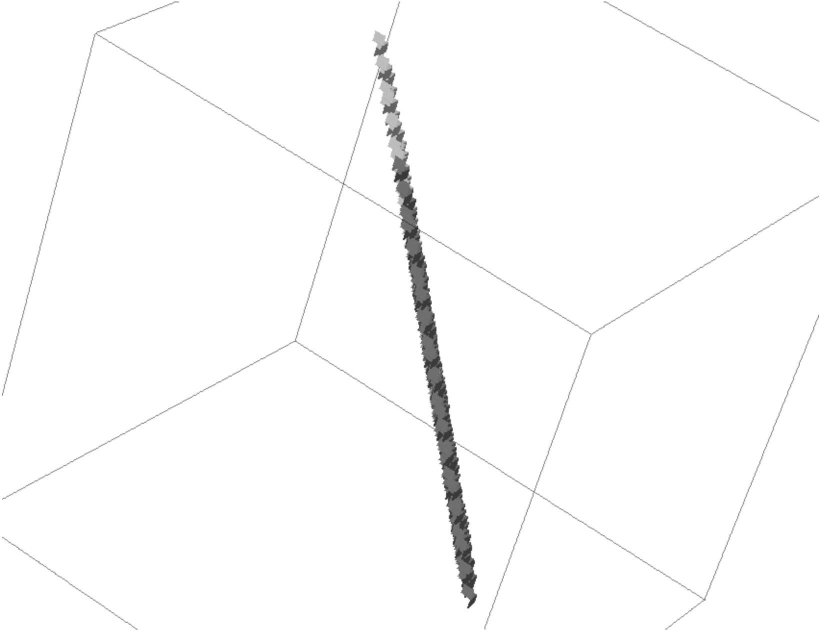






Definition of Rauzy fractals using $E_1^*(\sigma)$

Idea: **renormalize** $E_1^*(\sigma)^n($$)$ with $n \to \infty$.

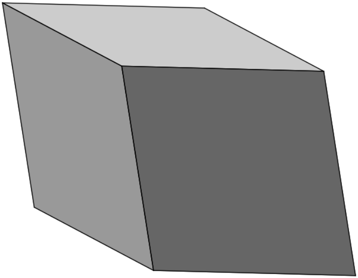
σ is Pisot:

- One expanding direction of $\mathbf{M}_\sigma$ (eigenvector $\mathbf{u}_\beta$)
- Two contracting directions of $\mathbf{M}_\sigma$ (eigenvectors $\mathbf{u}_{\beta'}$ and $\mathbf{u}_{\beta''}$)
- Let $\mathbb{P}_c$ be the contracting plane of $\mathbf{M}_\sigma$ spanned by $\mathbf{u}_{\beta'}, \mathbf{u}_{\beta''}$.
- Let $\pi : \mathbb{R}^3 \to \mathbb{P}_c$ be the projection on $\mathbb{P}_c$ along $\mathbf{u}_\beta$.
- **So:**

Renormalization $= \mathbf{M}_\sigma \circ \pi$

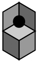
Definition of Rauzy fractals using $E_1^*(\sigma)$

$\pi($$)$

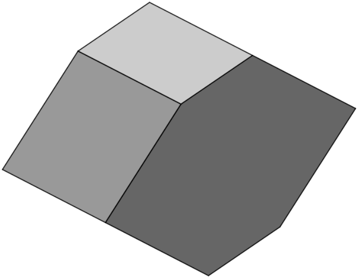


Definition of Rauzy fractals using $E_1^*(\sigma)$

$\mathbf{E}_1^*(\sigma)($  $)$

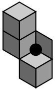


$\mathbf{M}_\sigma\pi(\mathbf{E}_1^*(\sigma)($  $))$

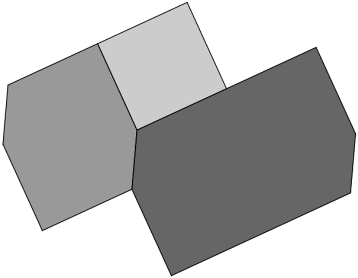


Definition of Rauzy fractals using $E_1^*(\sigma)$

$E_1^*(\sigma)^2(\;\;)$

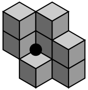


$M_\sigma^2\pi(E_1^*(\sigma)^2(\;\;))$

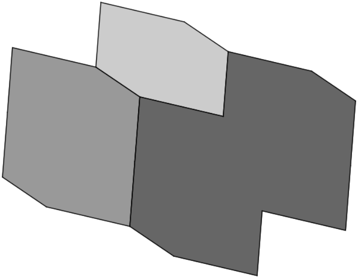


Definition of Rauzy fractals using $E_1^*(\sigma)$

$E_1^*(\sigma)^3(\,\cdot\,)$

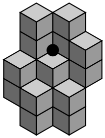


$M_\sigma^3 \pi(E_1^*(\sigma)^3(\,\cdot\,))$

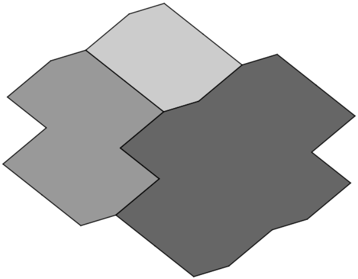


Definition of Rauzy fractals using $E_1^*(\sigma)$

$E_1^*(\sigma)^4(\text{cube})$

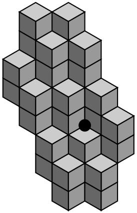


$M_\sigma^4 \pi(E_1^*(\sigma)^4(\text{cube}))$

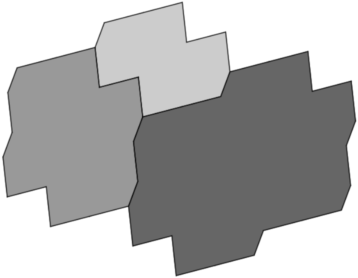


Definition of Rauzy fractals using $E_1^*(\sigma)$

$\mathbf{E}_1^*(\sigma)^5(\ )$

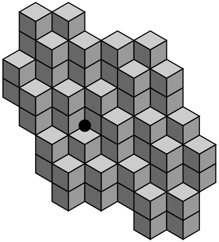


$\mathbf{M}_\sigma^5 \pi(\mathbf{E}_1^*(\sigma)^5(\ ))$

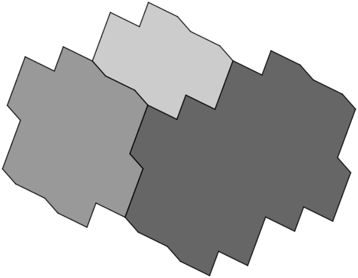


Definition of Rauzy fractals using $E_1^*(\sigma)$

$E_1^*(\sigma)^6($  $)$

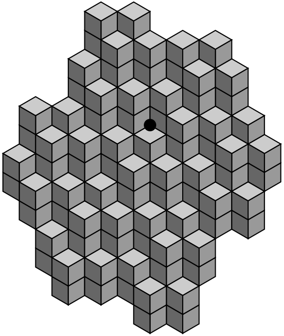


$M_\sigma^6 \pi(E_1^*(\sigma)^6($  $))$

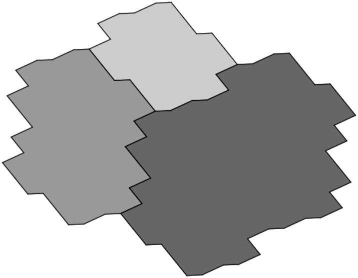


Definition of Rauzy fractals using $E_1^*(\sigma)$

$E_1^*(\sigma)^7($  $)$

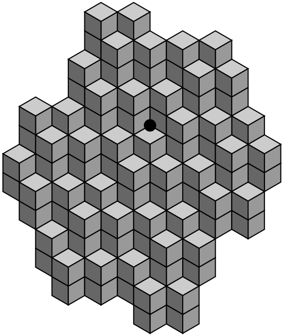


$M_\sigma^7 \pi(E_1^*(\sigma)^7($  $))$

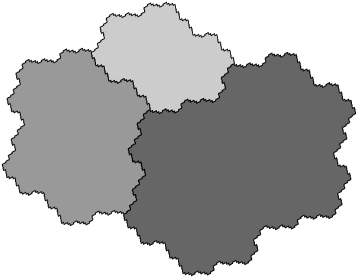


Definition of Rauzy fractals using $E_1^*(\sigma)$

$\mathbf{E}_1^*(\sigma)^7(\;)$



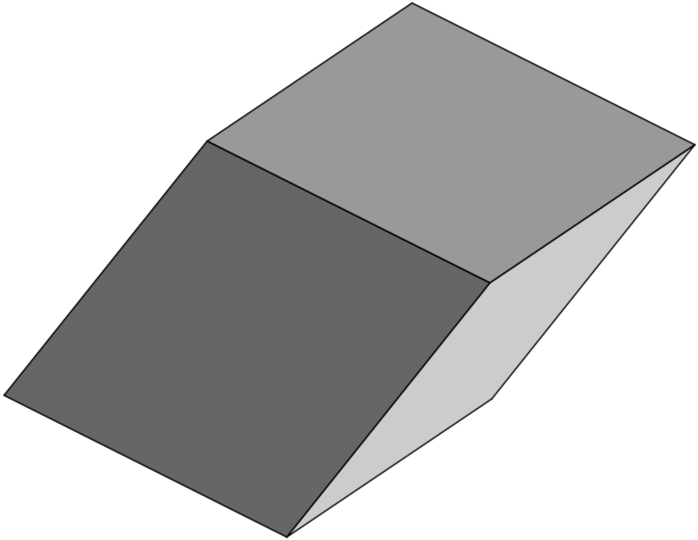
$\mathbf{M}_\sigma^\infty \pi(\mathbf{E}_1^*(\sigma)^\infty(\;))$



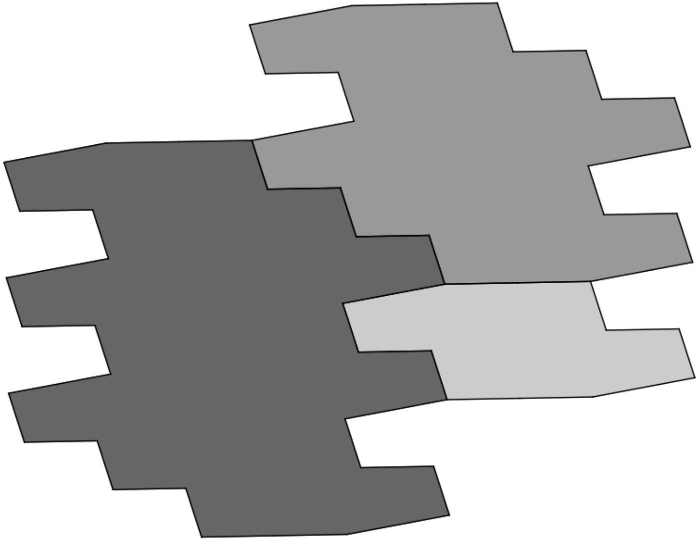
Definition [Arnoux-Ito 2001]

The **Rauzy fractal** of σ is the Hausdorf limit of $\mathbf{M}_\sigma^n \pi(\mathbf{E}_1^*(\sigma)^n(\;))$ as $n \to \infty$.

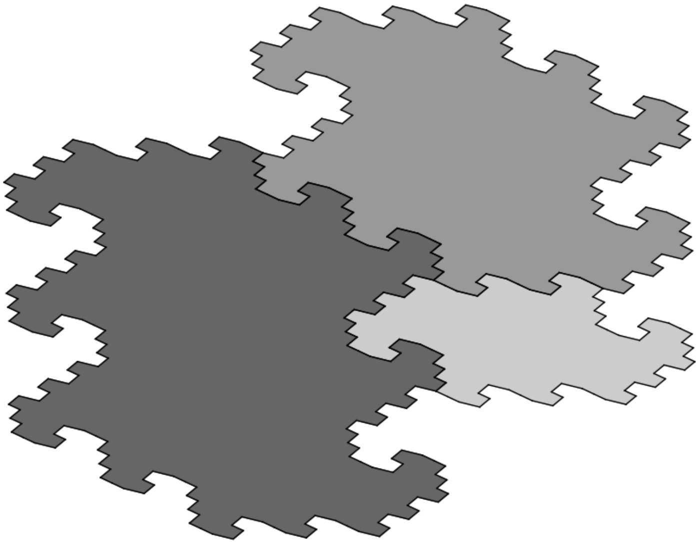
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



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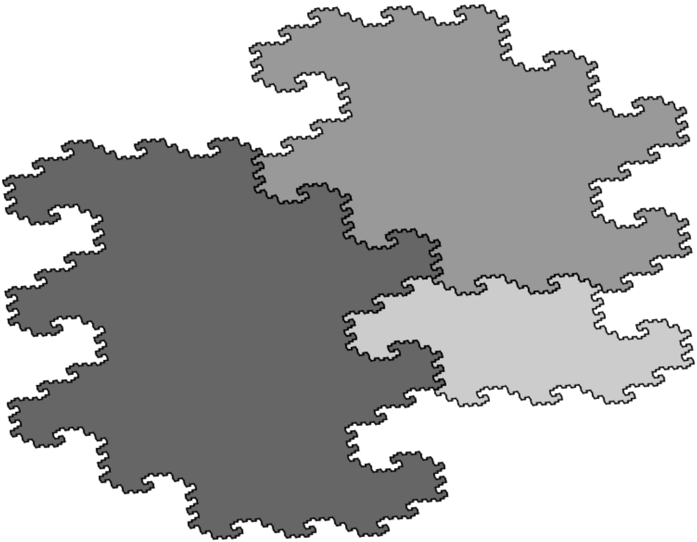
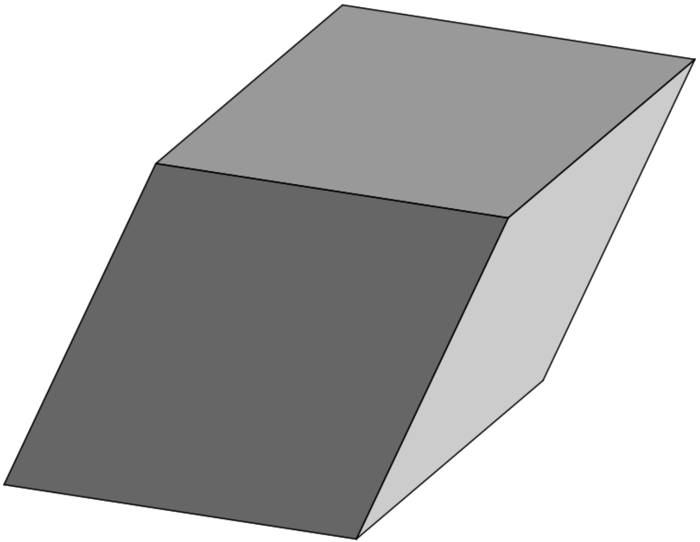


Rauzy fractal of $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



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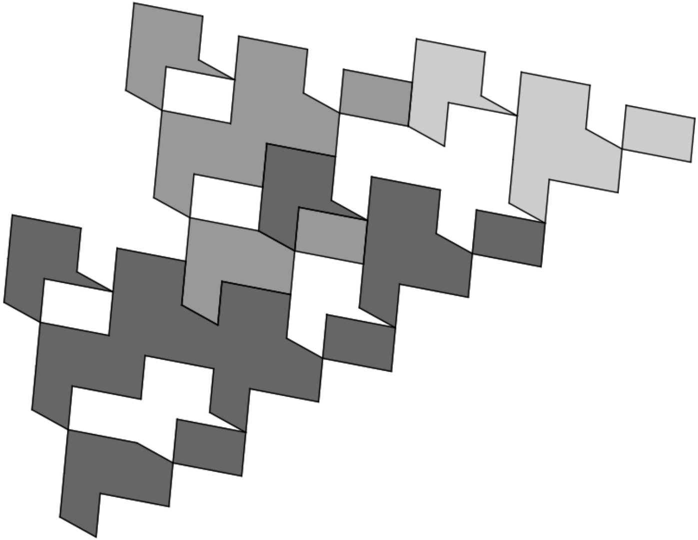

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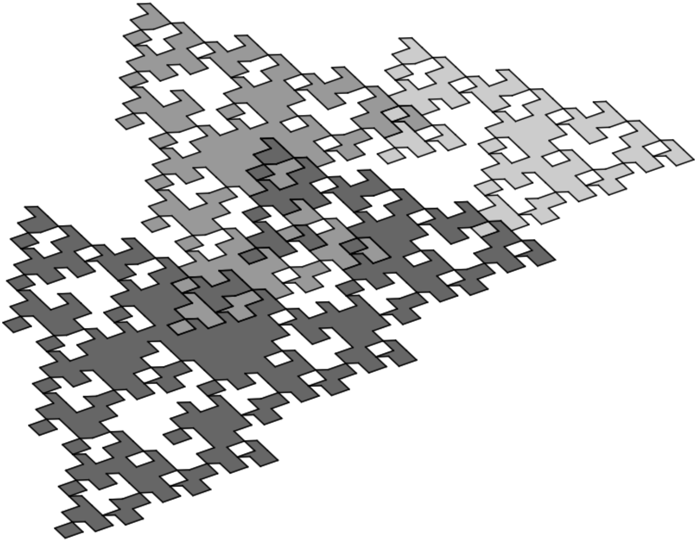

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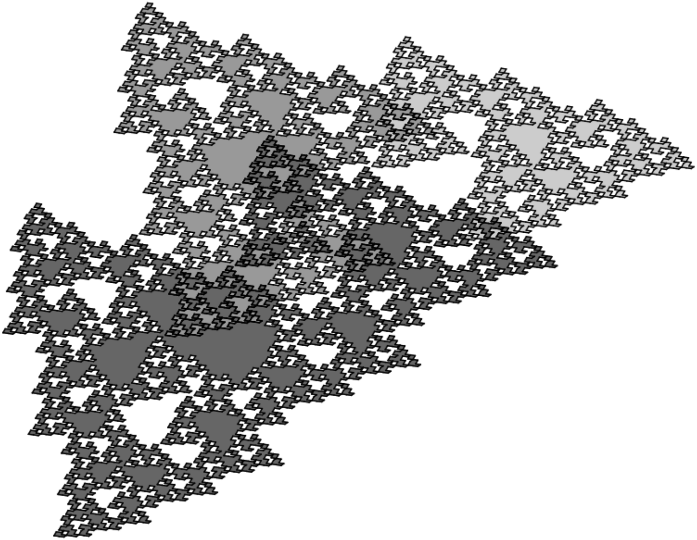

Combinatorial criterion

Theorem [Ito-Rao 2006]

Pisot conjecture holds for σ if and only if $\mathbf{E}_1^*(\sigma)^n([0, i]^*)$ contains arbitrarily large balls as $n \to \infty$, for $i \in \{1, 2, 3\}$.

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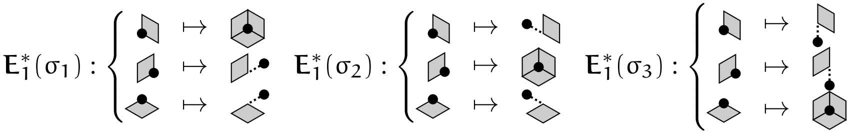
- How do we prove that balls grow in $\mathbf{E}_1^*(\sigma)^n([0,i]^*)$?
- We illustrate the technique with Arnoux-Rauzy substitutions:

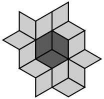
$$\sigma_1 : \begin{cases} 1 & \mapsto & 1 \\ 2 & \mapsto & 21 \\ 3 & \mapsto & 31 \end{cases} \qquad \sigma_2 : \begin{cases} 1 & \mapsto & 12 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 32 \end{cases} \qquad \sigma_3 : \begin{cases} 1 & \mapsto & 13 \\ 2 & \mapsto & 23 \\ 3 & \mapsto & 3 \end{cases}$$



The annulus property

Annulus A of P:

- A is edge-connected ("1-thick"),
- A "surrounds" P.



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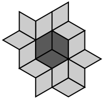

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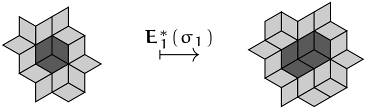

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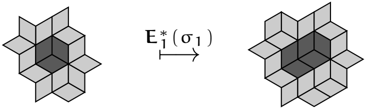


The annulus property: $E_1^*(\sigma)(\text{annulus}) = \text{annulus}$.

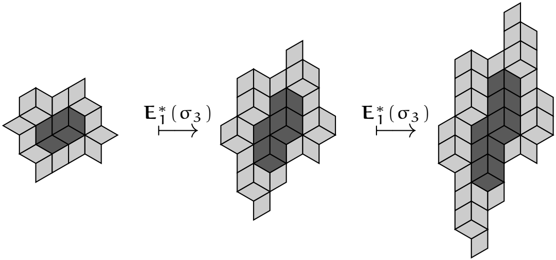
Proof idea, by induction: If

1. $E_1^*(\sigma)([\mathbf{0}, i]^*)^n$ contains an annulus for some n.
2. Annulus property for $E_1^*(\sigma)$.

Then $E_1^*(\sigma)([\mathbf{0}, i]^*)^n$ contains arbitrarily large balls. (induction)

Unfortunately. . .

. . . the annulus property **doesn't hold**:



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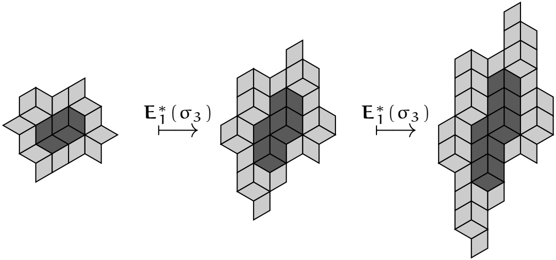

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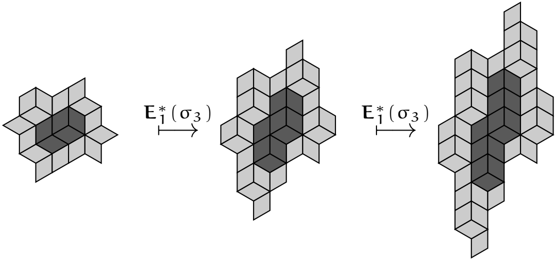
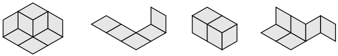
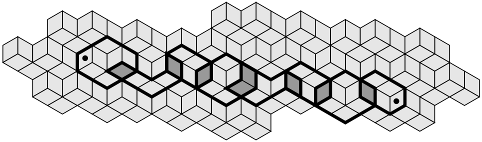


- ➥ We have to be more careful.
- ➥ Stronger assumptions on A: **covering properties**.

$\mathcal{L}$-coverings

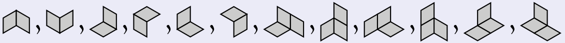
$\mathcal{L} = \Big\{$  $\Big\}$



Definition

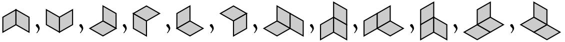
P is **$\mathcal{L}$-covered** if for all $f, g \in P$, there is an **$\mathcal{L}$-path from f to g**.

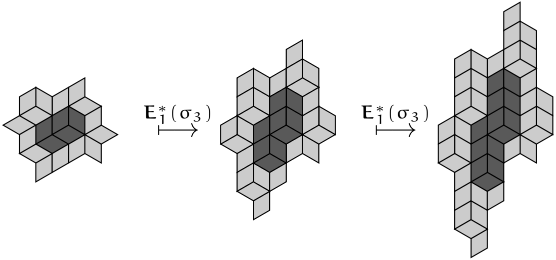
Proposition

Let $\mathcal{L}_{AR} = \{$  $\}$

P is $\mathcal{L}_{AR}$-covered $\Rightarrow$ $\mathbf{E}_1^*(\sigma_i)(P)$ is $\mathcal{L}_{AR}$-covered, for all $i \in \{1, 2, 3\}$.

$\mathcal{L}$-coverings and Arnoux-Rauzy

- Let $\mathcal{L}_{\mathrm{AR}} = \{$  $\}$
- Let's look at our problem again:



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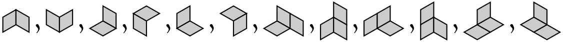
- Let $\mathcal{L}_{\mathrm{AR}} = \{$  $\}$
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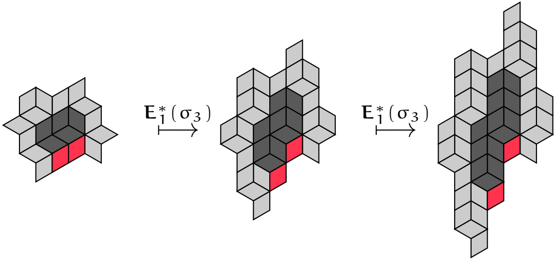

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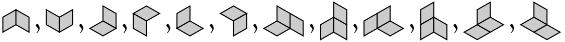
- Let $\mathcal{L}_{AR} = \{$  $\}$
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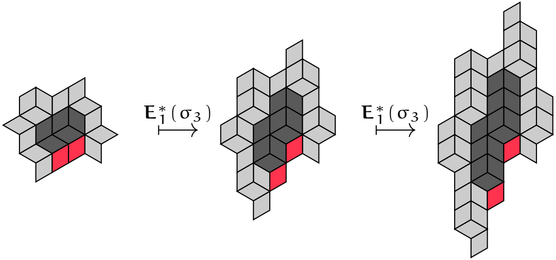

- The annulus **is** $\mathcal{L}_{AR}$-covered, but something is missing:

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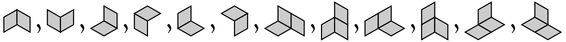
- Let $\mathcal{L}_{\mathsf{AR}} = \{$  $\}$
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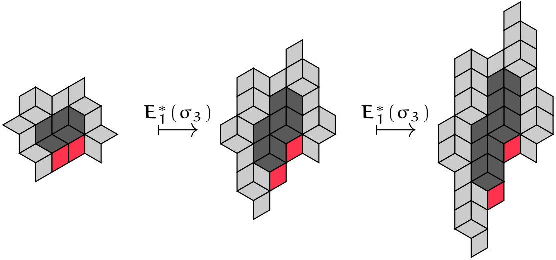

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Definition

$\mathcal{A}$ is **strongly $\mathcal{L}_{\mathsf{AR}}$-covered** if it is $\mathcal{L}_{\mathsf{AR}}$ covered,
and if for every two-face edge-connected pattern X,
there exists $Y \in \mathcal{L}_{\mathsf{AR}}$ such that $X \subseteq Y \subseteq \mathcal{A}$.

$\mathcal{L}$-coverings and Arnoux-Rauzy

Proposition (Annulus Property, induction step)

If an annulus A is strongly $\mathcal{L}_{\mathsf{AR}}$-covered,
then $\mathbf{E}_1^*(\sigma_i)(A)$ is a strongly $\mathcal{L}_{\mathsf{AR}}$-covered annulus.

$\mathcal{L}$-coverings and Arnoux-Rauzy

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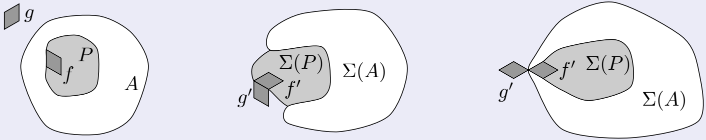

- Enumerate all two-face connected patterns $f' \cup g'$ with disconnected preimage $f \cup g$.

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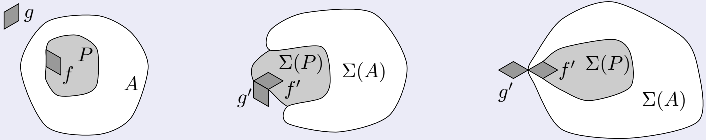

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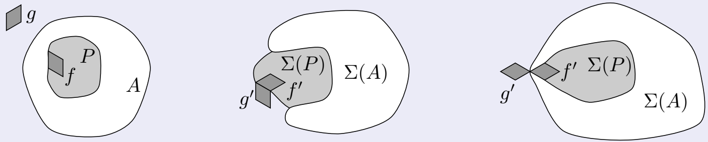

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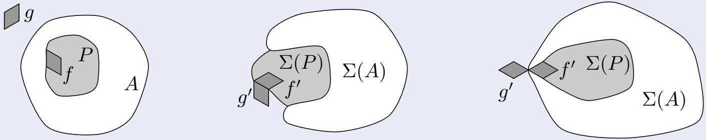
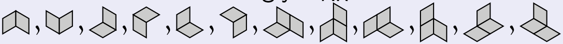


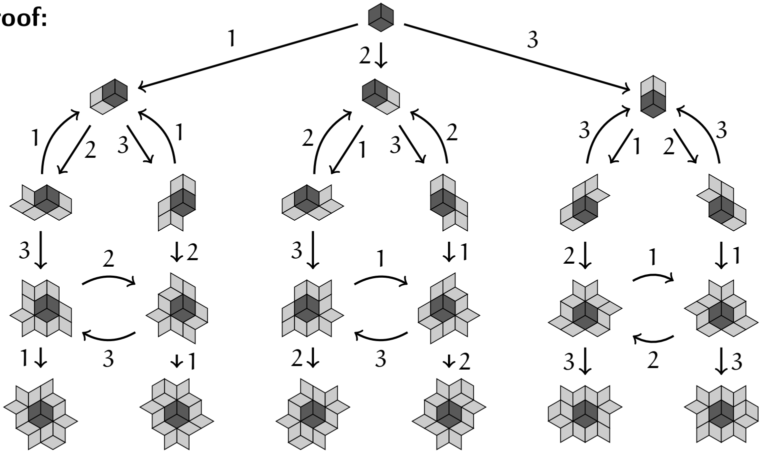
- Enumerate all two-face connected patterns $f' \cup g'$ with disconnected preimage $f \cup g$.
- Example: $f \cup g \;=$  with  $\subseteq A$ the only possible completion
- **Contradiction** because A is strongly $\mathcal{L}_{\mathsf{AR}}$-covered.
  $\mathcal{L}_{\mathsf{AR}} \;=\; \{$  $\}$

$\mathcal{L}$-coverings and Arnoux-Rauzy

Proposition (Annulus Property, induction initialization)

If σ is a product of σ_i with at least one of each σ_i,
then $\mathbf{E}_1^*(\sigma^2)($$)$ contains a strongly $\mathcal{L}_{\mathrm{AR}}$-covered annulus.

Proof:



Arnoux-Rauzy finite products

Theorem [Berthé-J.-Siegel 2012]

The super coincidence condition holds for every finite product σ of Arnoux-Rauzy substitutions (in which each σ_i appears at least once).

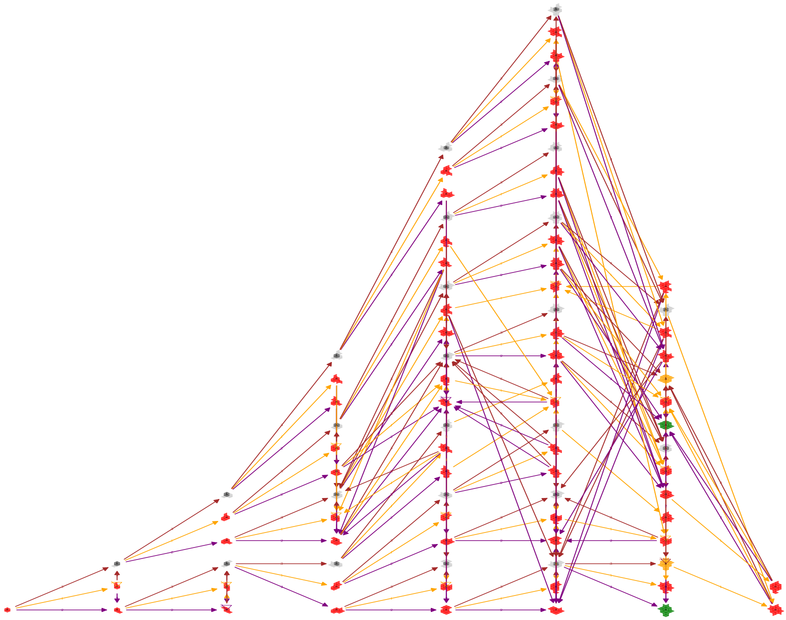
Other consquences:

- The Rauzy fractal of σ is connected.
- The origin is an inner point of the Rauzy fractal.

Other substitutions?

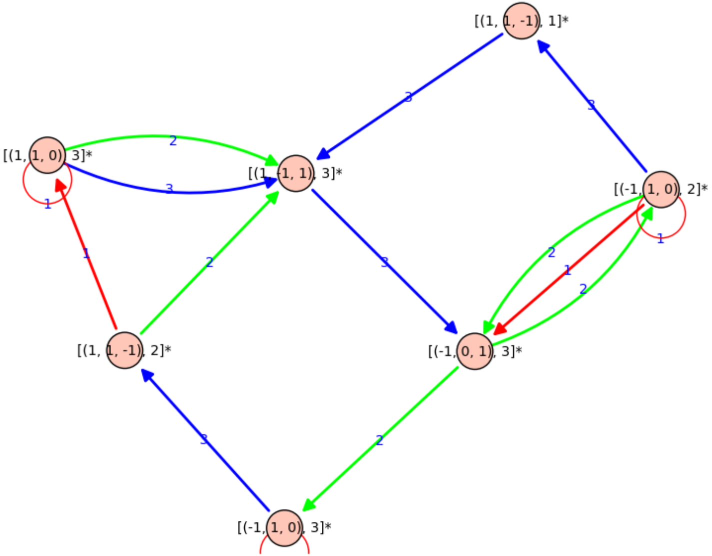
- The induction step can be proved similarly for other families (Jacobi-Perron, Brun, ...) [Work in progress, 2011-2012]
- The initialization step is **more difficult**: sometimes the balls don't grow around 0...

Jacobi-Perron: complicated study



Brun

Characterization of products for which zero is an inner point of the fractal:



Merci de votre attention.