Rauzy fractals associated with cubic real number fields

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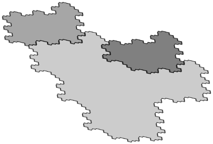
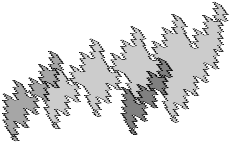
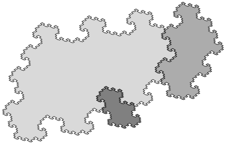
Joint work with Valérie Berthé and Anne Siegel

Algebra Seminar
Technische Universität Wien
Freitag, den 2. Dezember 2011

The central tool: substitutions

$$\begin{aligned} 1 &\mapsto 12 \\ 2 &\mapsto 1 \end{aligned}$$

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$$\begin{aligned} 1 &\mapsto 12 \\ 2 &\mapsto 1 \end{aligned} \qquad 1$$

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12

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$$\begin{aligned} 1 &\mapsto 12 \\ 2 &\mapsto 1 \end{aligned} \qquad 121$$

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$$1211212112112121121 21$$

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$$\begin{aligned} 1 &\mapsto 12 \\ 2 &\mapsto 1 \end{aligned}$$

$1211212112112121121211211212112112$

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$$\begin{array}{ccc} 1 & \mapsto & 12 \\ 2 & \mapsto & 1 \end{array}$$

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$$\begin{aligned} 1 &\mapsto 12 \\ 2 &\mapsto 1 \end{aligned}$$

121121211211212112121121121211211212112121121

$$\begin{aligned} a &\mapsto aabb \\ b &\mapsto aabb \end{aligned}$$

$$\begin{aligned} 0 &\mapsto 10 \\ 1 &\mapsto 01 \end{aligned}$$

$$\begin{aligned} x &\mapsto xxyz \\ y &\mapsto yz \\ z &\mapsto xyz \end{aligned}$$

$$\begin{aligned} 1 &\mapsto 12 \\ 2 &\mapsto 13 \\ 3 &\mapsto 14 \\ 4 &\mapsto 1 \end{aligned}$$

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a

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$$\begin{aligned} 0 &\mapsto 10 \\ 1 &\mapsto 01 \end{aligned} \qquad 10$$

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$$\begin{aligned} 1 &\mapsto 12 \\ 2 &\mapsto 13 \\ 3 &\mapsto 14 \\ 4 &\mapsto 1 \end{aligned} \qquad 1213141$$

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$$\begin{aligned} 1 &\mapsto 12 \\ 2 &\mapsto 1 \end{aligned} \qquad 1211212112112121121211211212112121121$$

$$\begin{aligned} a &\mapsto aabb \\ b &\mapsto aabb \end{aligned} \qquad aabbaabbaabbaabb$$

$$\begin{aligned} 0 &\mapsto 10 \\ 1 &\mapsto 01 \end{aligned} \qquad 0110$$

$$\begin{aligned} x &\mapsto xxyz \\ y &\mapsto yz \\ z &\mapsto xyz \end{aligned} \qquad xxyzxxyzyzxyz$$

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10010110

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121312141213121121312121 3

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121121211211212112121121211212112121121

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0110100110010110

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➥ Symbolic dynamical system (X_σ, S). (Subshift of $\{1, 2, 3\}^{\mathbb{Z}}$.)

Rauzy fractals

To a substitution $\sigma : \{1,2,3\}^* \to \{1,2,3\}^*$, we can associate a **Rauzy fractal** (definition later).

- Compact subset of $\mathbb{R}^2$ with fractal boundary.
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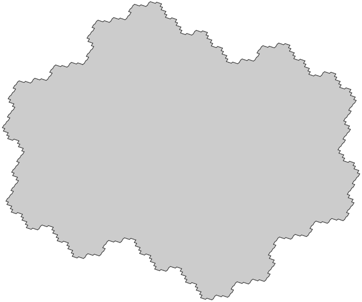

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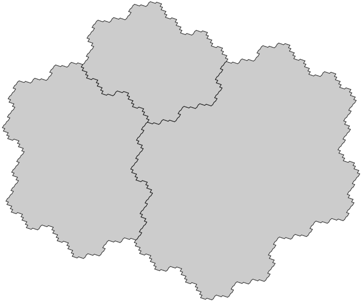

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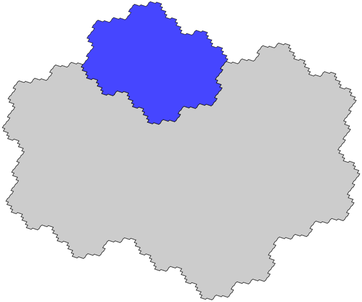

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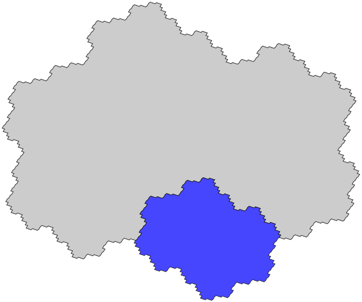

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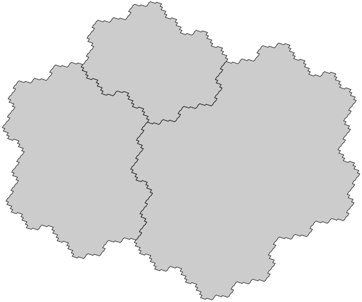

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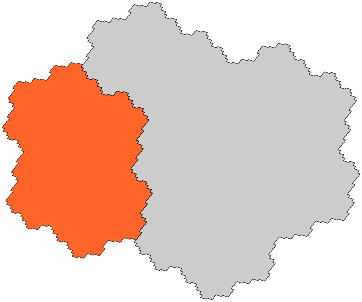

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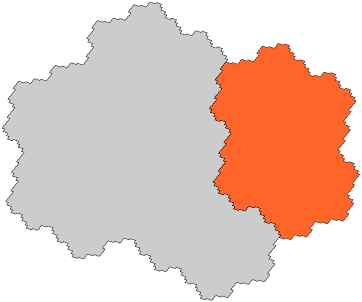

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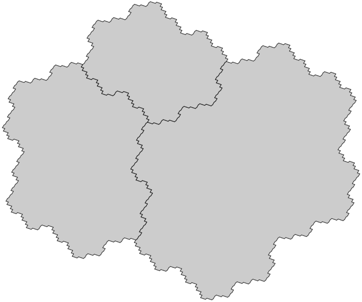

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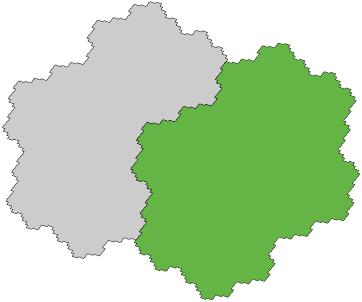

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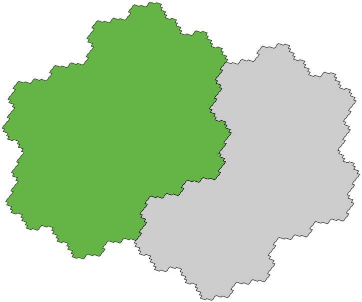

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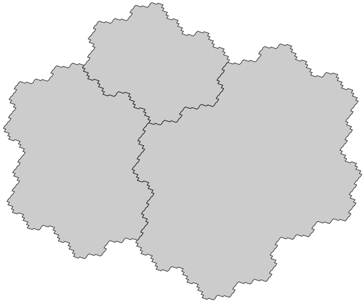

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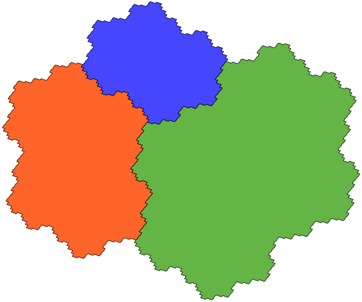

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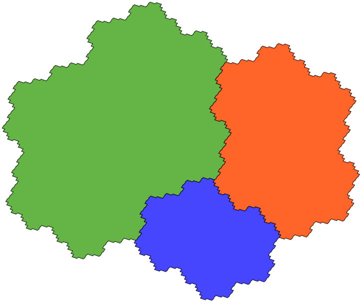
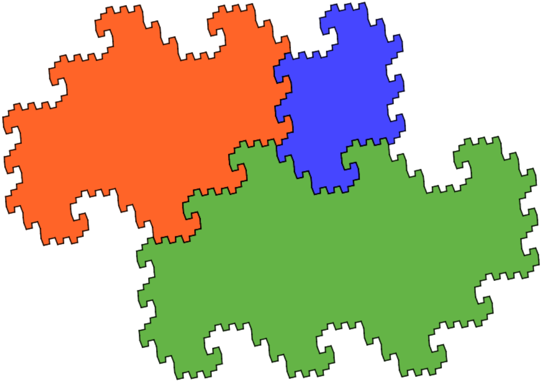
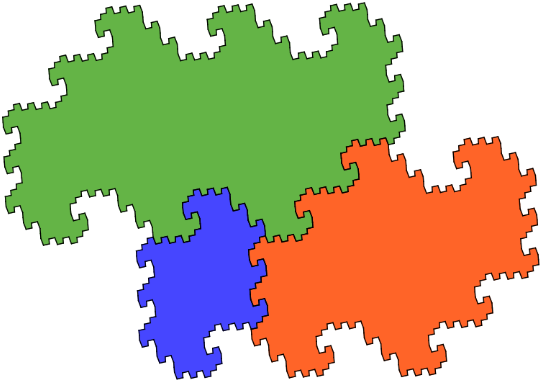


Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$



Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$



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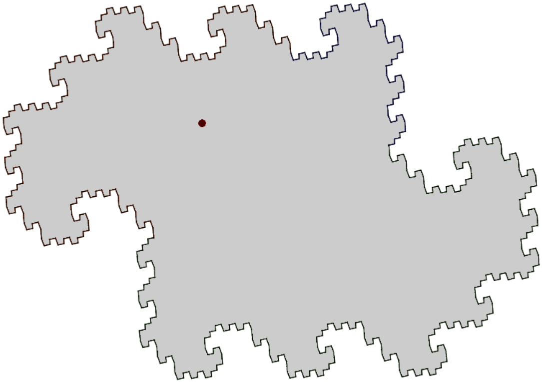

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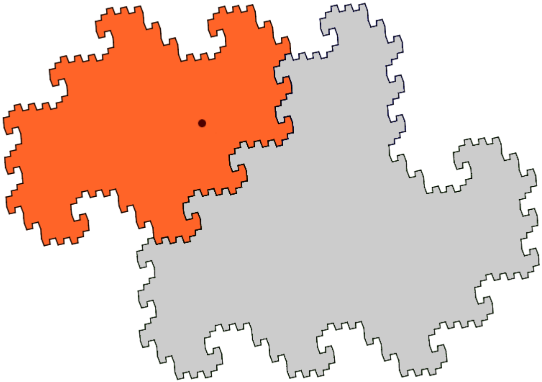

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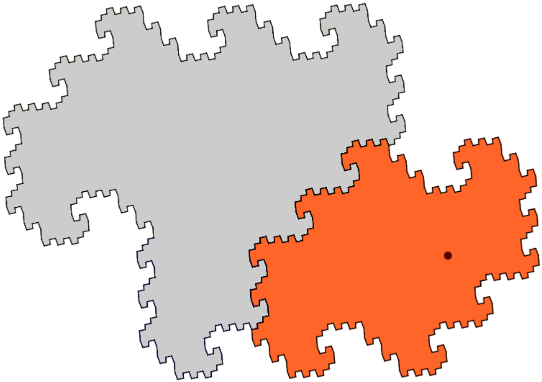

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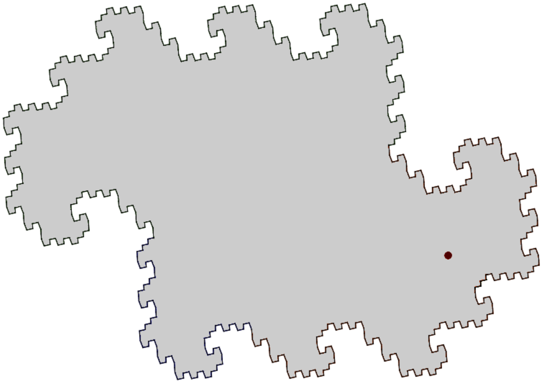
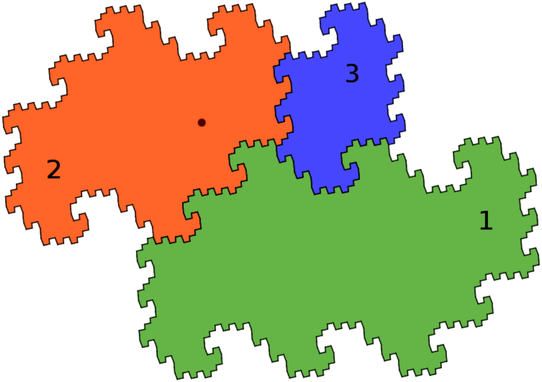


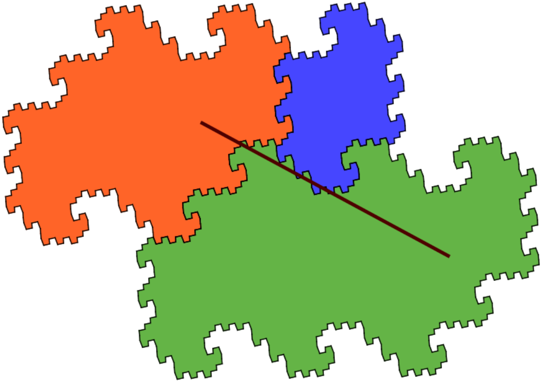
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 2



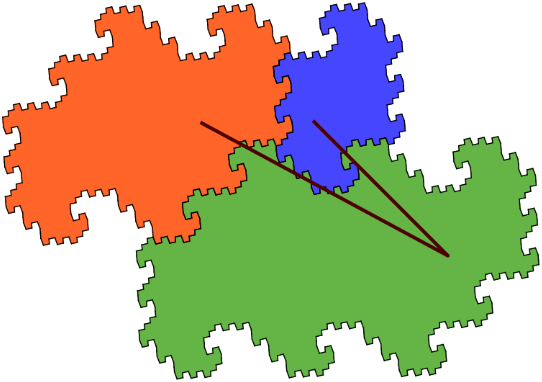
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 21



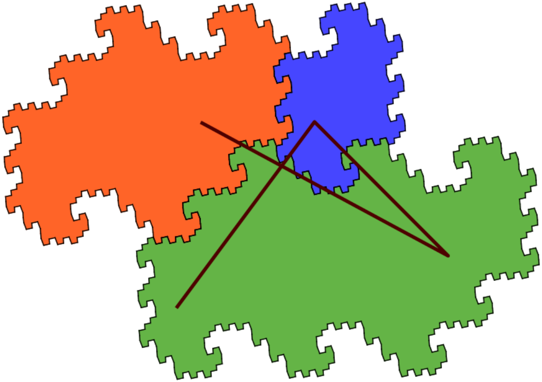
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 213



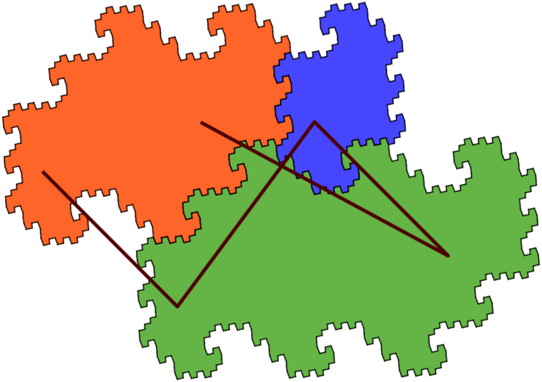
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 2131



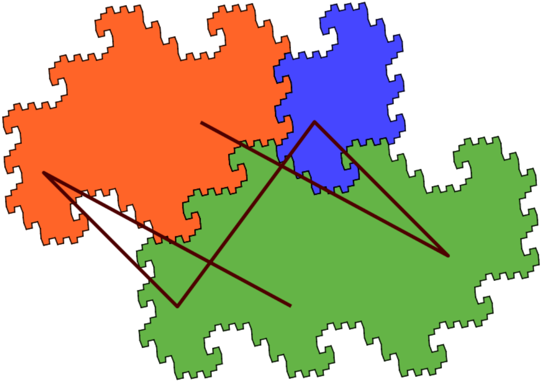
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 21312



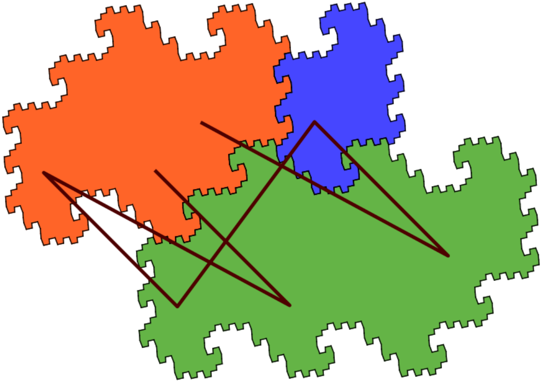
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 213121



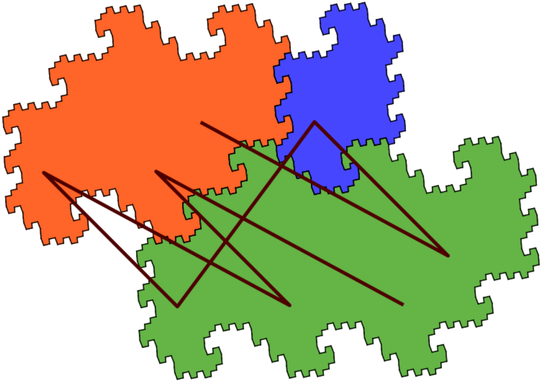
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 2131212



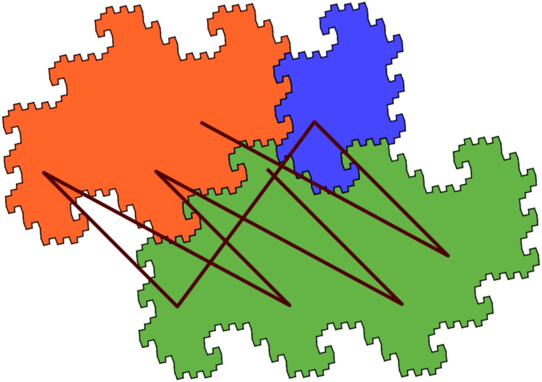
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 21312121



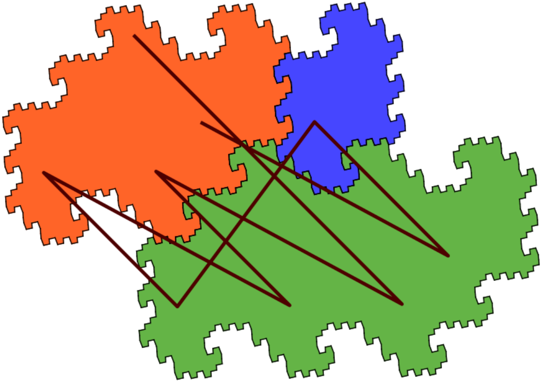
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 213121211



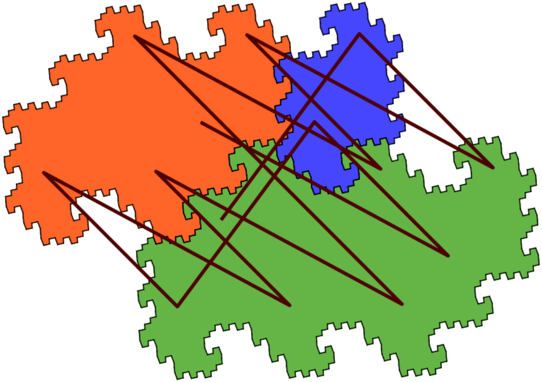
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 2131212112



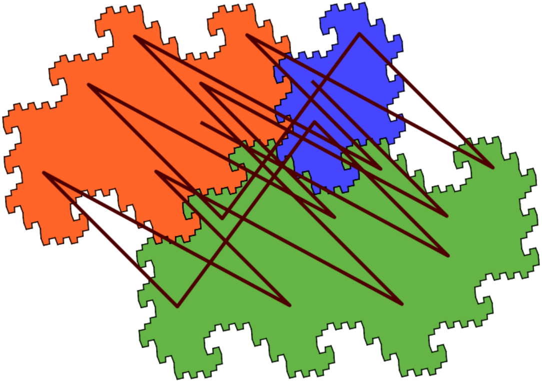
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 213121211212131



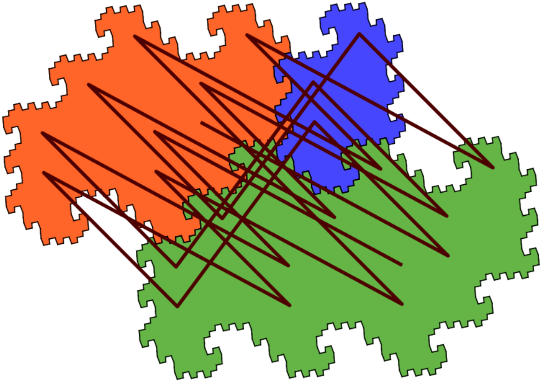
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 21312121121213121213



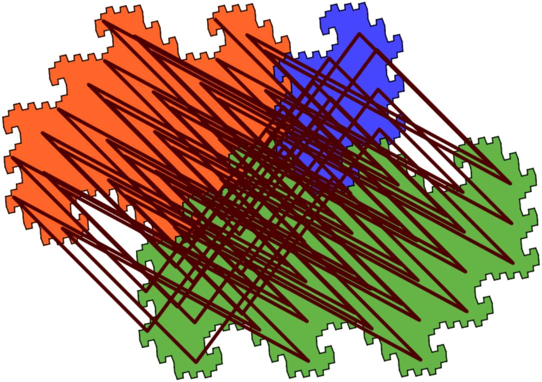
Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: 2131212112121312121312121



Dynamics of $\sigma : 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$

Orbit: $\cdots$2131212112121312121312121$\cdots \in X_\sigma \subseteq \{1,2,3\}^{\mathbb{Z}}$



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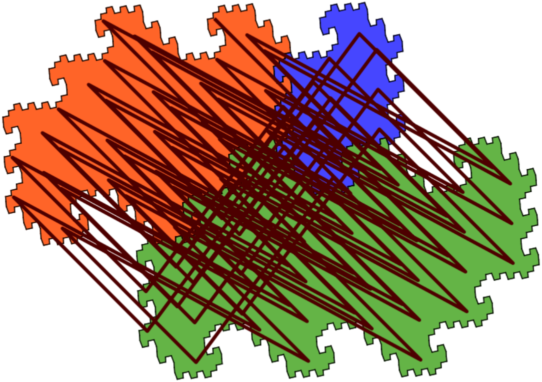
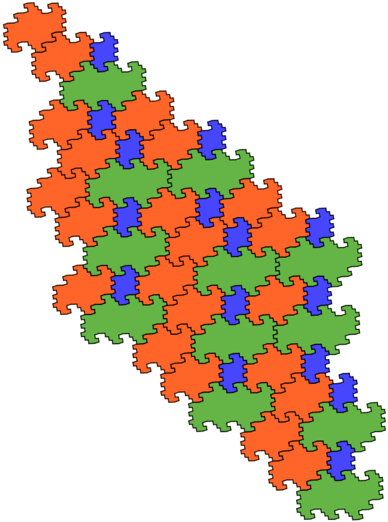


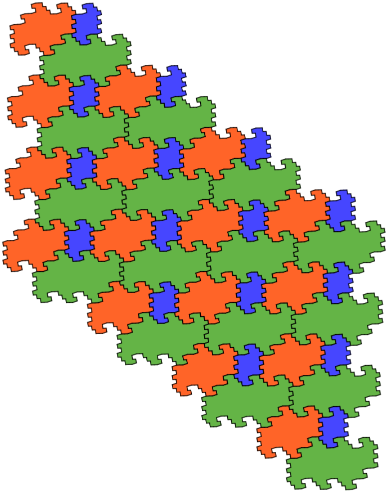
$\big(X_\sigma$, shift$\big)$ $\cong$ $\big($  , domain exchange$\big)$

Rauzy fractals: tilings of the plane

Self-similar (aperiodic) tiling:

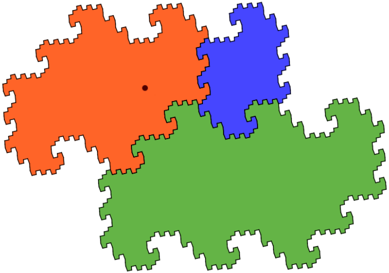


Periodic tiling:

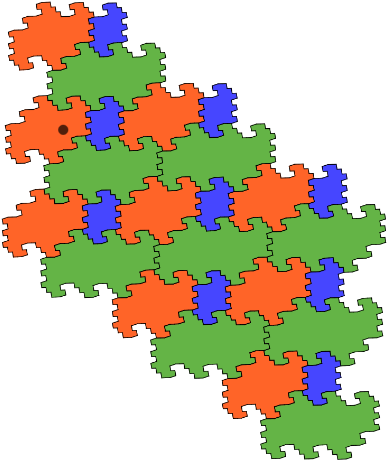


Dynamics of σ, continued

Domain exchange:



Translation on the torus:

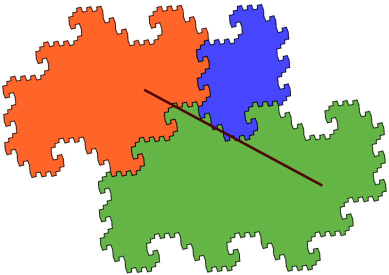


Shift:

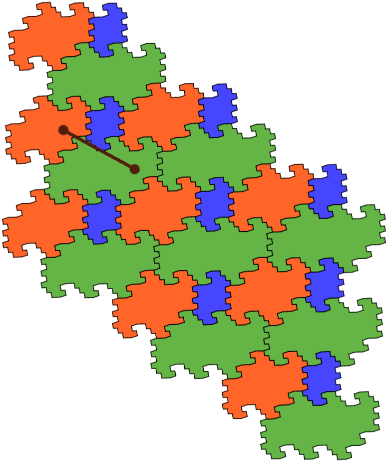
$$\cdots \underline{2}131212112 \cdots \in X_\sigma$$

Dynamics of σ, continued

Domain exchange:



Translation on the torus:

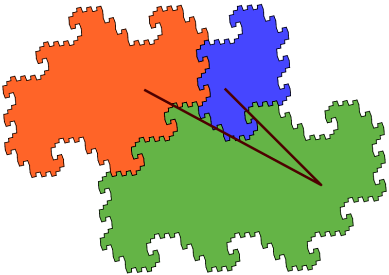


Shift:

$$\cdots 2\underline{1}31212112 \cdots \in X_\sigma$$

Dynamics of σ, continued

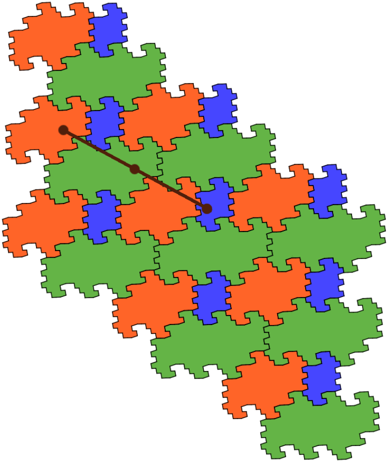
Domain exchange:



Shift:

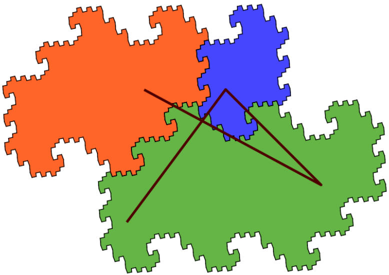
$$\cdots 21\underline{3}1212112 \cdots \in X_\sigma$$

Translation on the torus:

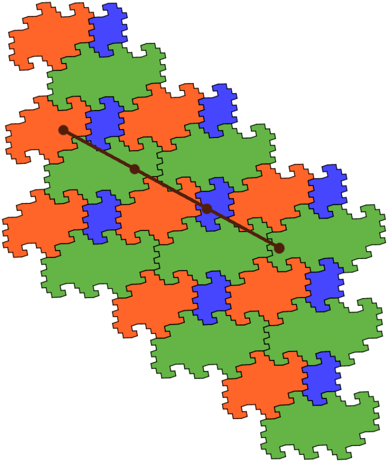


Dynamics of σ, continued

Domain exchange:



Translation on the torus:

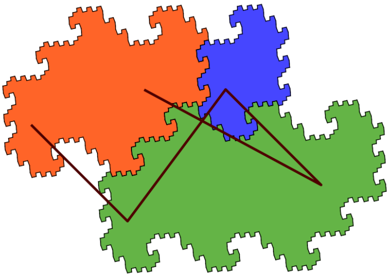


Shift:

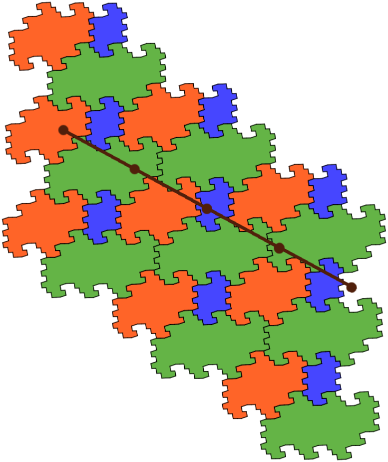
$$\cdots 2131\underline{1}212112 \cdots \in X_\sigma$$

Dynamics of σ, continued

Domain exchange:



Translation on the torus:

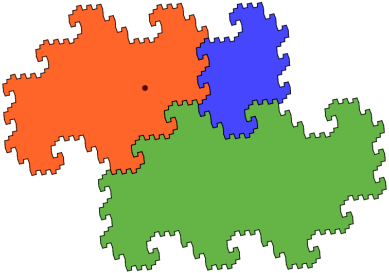


Shift:

$$\cdots 2131\underline{2}12112 \cdots \in X_\sigma$$

Dynamics of σ, continued

Domain exchange:



Shift:

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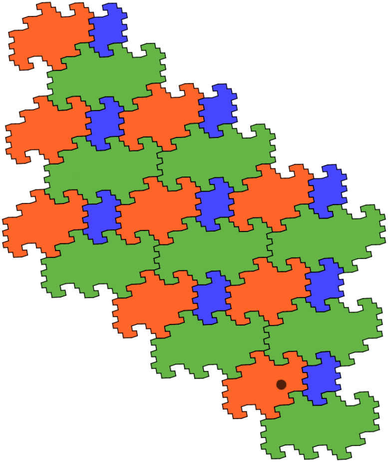
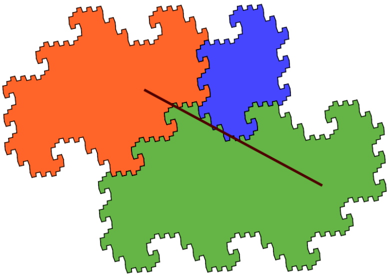
Translation on the torus:

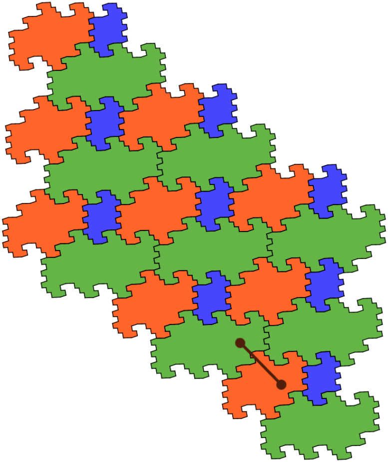


Dynamics of σ, continued

Domain exchange:



Translation on the torus:



Shift:

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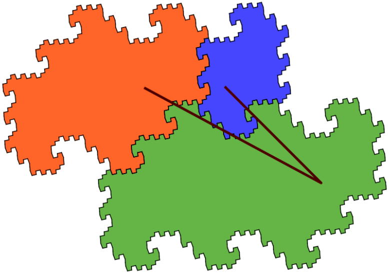
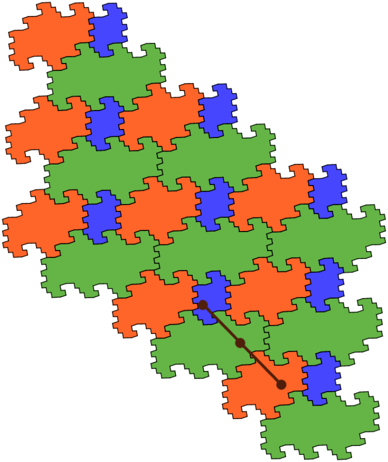
Dynamics of σ, continued

Domain exchange:



Translation on the torus:



Shift:

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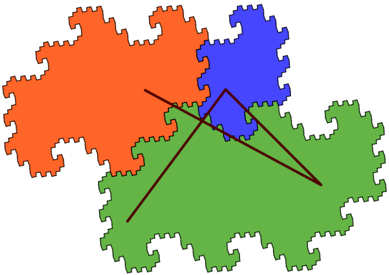
Dynamics of σ, continued

Domain exchange:



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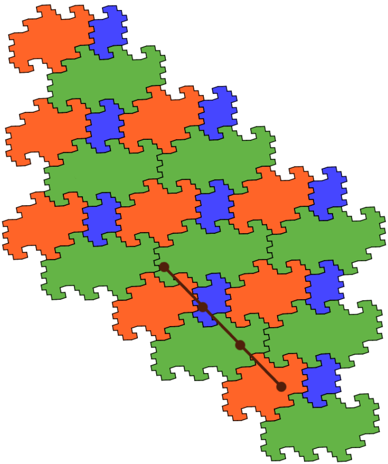

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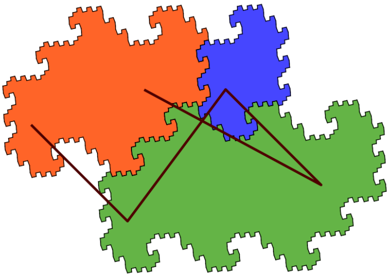
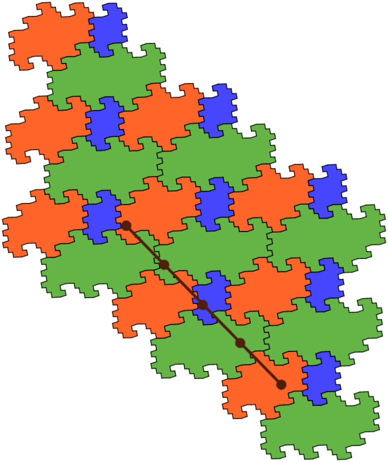
Dynamics of σ, continued

Domain exchange:



Translation on the torus:



Shift:

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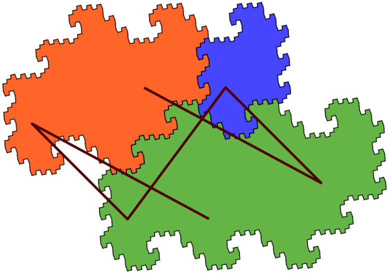
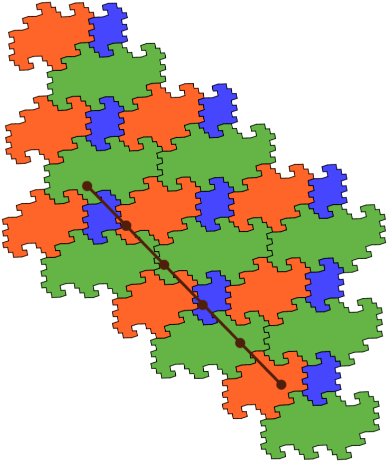
Dynamics of σ, continued

Domain exchange:



Translation on the torus:

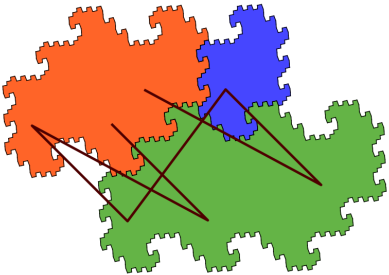


Shift:

$$\cdots 2131 2\underline{1} 2112 \cdots \in X_\sigma$$

Dynamics of σ, continued

Domain exchange:



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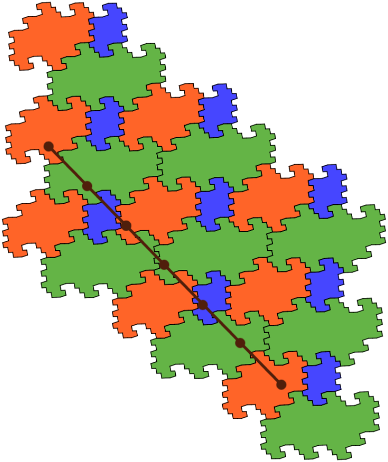

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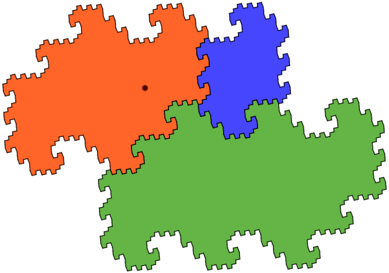
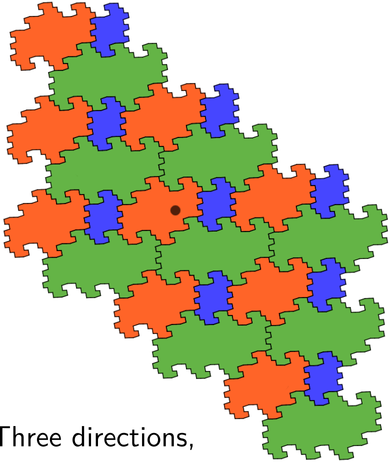
$$\cdots 2131212\underline{2}112 \cdots \in X_\sigma$$

Dynamics of σ, continued

Domain exchange:



Translation on the torus:



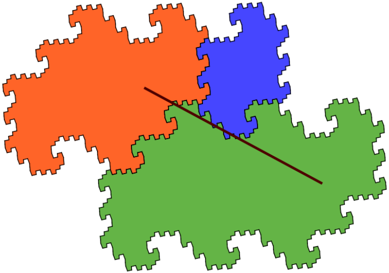
Shift:

$$\cdots \underline{2}131212112 \cdots \in X_\sigma$$

Three directions,
same translation.

Dynamics of σ, continued

Domain exchange:



Shift:

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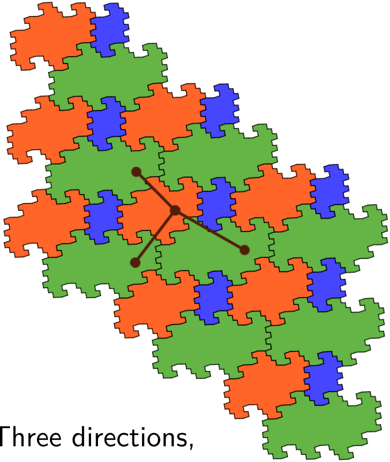
Translation on the torus:



Three directions,
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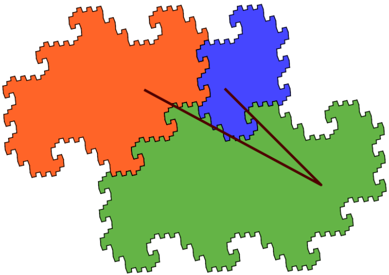
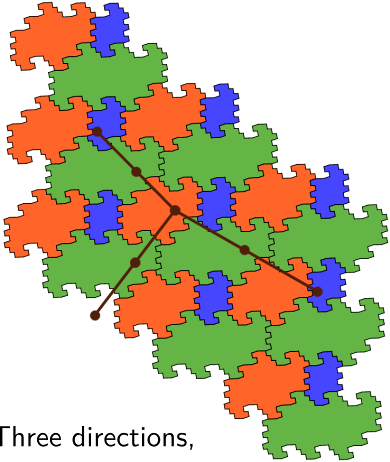
Dynamics of σ, continued

Domain exchange:



Translation on the torus:



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Three directions,
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Dynamics of σ, summary

$$\Big(X_\sigma\,,\text{shift}\Big) \;\cong\; \Big($$  $$,\text{domain exchange}\Big) \;\cong\; \Big(\mathbb{T}^2\,,\text{translation}\Big)$$

Dynamics of σ, summary

$\left(X_\sigma\,, \text{shift}\right) \;\cong\; \Big($  $, \text{domain exchange}\Big) \;\cong\; \left(\mathbb{T}^2\,, \text{translation}\right)$

Are these nice properties always true?

Dynamics of σ, summary

$\left(X_\sigma\,,\text{shift}\right) \;\cong\; \left(\right.$  $\left.,\text{domain exchange}\right) \;\cong\; \left(\mathbb{T}^2\,,\text{translation}\right)$

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Pisot conjecture

Yes, if σ is unimodular Pisot irreducible.

Dynamics of σ, summary

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Are these nice properties always true?

Pisot conjecture

Yes, if σ is unimodular Pisot irreducible.

Equivalent formulations:

- The tiles don't overlap in the tilings.
- Pure discreteness of the spectrum of (X_σ, S).
- *Coincidence conditions* on σ.
- Geometric combinatorial conditions (*cf.* later).
- Many criteria, from many different viewpoints.

Today we would like to prove. . .

Main result

Theorem [Berthé-J.-Siegel 2011]

Let $\mathbb{K}$ be a cubic real extension of $\mathbb{Q}$.
There exist $\alpha, \beta \in \mathbb{K}$ and an unimodular Pisot irreducible substitution σ such that:

1. $\mathbb{K} = \mathbb{Q}(\alpha, \beta)$.
2. $(\mathbb{T}^2, T_{\alpha,\beta}) \cong (X_\sigma, S)$.

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1. Explain where (α, β) and σ come from.
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In this talk, we will:

1. Explain where (α, β) and σ come from.
   (Jacobi-Perron algorithm and Dubois-Paysant Leroux's result)
2. Prove the isomorphism.
   (geometric combinatorial methods)

1. **Find α, β.**

- **Jacobi-Perron algorithm**
- **A result of Dubois and Paysant-Le Roux**

Jacobi-Perron algorithm

Let $\mathbf{v} = (a,\ b,\ c) \in \mathbb{R}^3$ be $\mathbb{Q}$-linearly independent, $0 < a, b \leqslant c$.

JP algorithm

$$\mathbf{v} = (a, b, c) \quad \overset{\text{JP}}{\longmapsto} \quad \mathbf{v}_1 = (b - \lfloor b/a \rfloor a,\ c - \lfloor c/a \rfloor a,\ a)$$

Jacobi-Perron algorithm

Let $\mathbf{v} = (a,\ b,\ c) \in \mathbb{R}^3$ be $\mathbb{Q}$-linearly independent, $0 < a, b \leqslant c$.

JP algorithm

$$\mathbf{v} = (a, b, c) \quad \overset{\text{JP}}{\longmapsto} \quad \mathbf{v}_1 = (b - \lfloor b/a \rfloor a,\ c - \lfloor c/a \rfloor a,\ a)$$

- Generalizes continued fraction expansions (Euclid's algorithm). (Continued fraction of $\alpha \iff$ JP expansion of $(1, \alpha)$.)
- Yields simultaneous rationnal approximations of a, b, c.
- Good convergence properties.
- Many other generalizations exist! (Brun, Poincaré, Selmer, Arnoux-Rauzy, Tamura-Yasutomi, . . .)

Example

$$\mathbf{v} = (1,\ \ \sqrt{2},\ \ \sqrt{\pi}) \quad \mapsto \quad (\sqrt{2} - 1,\ \ \sqrt{\pi} - 1,\ 1)$$

$$B_1 = \lfloor \tfrac{\sqrt{2}}{1} \rfloor = 1 \qquad\qquad C_1 = \lfloor \tfrac{\sqrt{\pi}}{1} \rfloor = 1$$

Example

$$\begin{aligned}\mathbf{v} = (1,\ \sqrt{2},\ \sqrt{\pi}) \ &\mapsto\ (\sqrt{2}-1,\ \sqrt{\pi}-1,\ 1) \\ &\mapsto\ (\sqrt{\pi}-\sqrt{2},\ 3-2\sqrt{2},\ \sqrt{2}-1)\end{aligned}$$

$$B_1 = \lfloor \tfrac{\sqrt{2}}{1} \rfloor = 1 \qquad C_1 = \lfloor \tfrac{\sqrt{\pi}}{1} \rfloor = 1$$

$$B_2 = \lfloor \tfrac{\sqrt{\pi}-1}{\sqrt{2}-1} \rfloor = 1 \qquad C_2 = \lfloor \tfrac{1}{\sqrt{2}-1} \rfloor = 2$$

Example

$$\begin{aligned}\mathbf{v} = (1,\ \sqrt{2},\ \sqrt{\pi}) &\mapsto (\sqrt{2}-1,\ \sqrt{\pi}-1,\ 1)\\ &\mapsto (\sqrt{\pi}-\sqrt{2},\ 3-2\sqrt{2},\ \sqrt{2}-1)\\ &\mapsto \cdots\end{aligned}$$

$$\begin{aligned} B_1 &= \lfloor \tfrac{\sqrt{2}}{1} \rfloor = 1 & C_1 &= \lfloor \tfrac{\sqrt{\pi}}{1} \rfloor = 1\\ B_2 &= \lfloor \tfrac{\sqrt{\pi}-1}{\sqrt{2}-1} \rfloor = 1 & C_2 &= \lfloor \tfrac{1}{\sqrt{2}-1} \rfloor = 2\\ B_3 &= \lfloor \tfrac{3-2\sqrt{2}}{\sqrt{\pi}-\sqrt{2}} \rfloor = 0 & C_3 &= \lfloor \tfrac{\sqrt{2}-1}{\sqrt{\pi}-\sqrt{2}} \rfloor = 1\end{aligned}$$

Example

$$\mathbf{v} = (1,\ \sqrt{2},\ \sqrt{\pi}) \mapsto (\sqrt{2}-1,\ \sqrt{\pi}-1,\ 1)$$
$$\mapsto (\sqrt{\pi}-\sqrt{2},\ 3-2\sqrt{2},\ \sqrt{2}-1)$$
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$$B_3 = \lfloor \tfrac{3-2\sqrt{2}}{\sqrt{\pi}-\sqrt{2}} \rfloor = 0 \qquad C_3 = \lfloor \tfrac{\sqrt{2}-1}{\sqrt{\pi}-\sqrt{2}} \rfloor = 1$$

Expansion $(B_n, C_n)_{n\geqslant 1}$:
$(1,1),(1,2),(0,1),(0,2),(0,3),(0,3),(2,4),(0,1),(0,7),$
$(0,1),(29,36),(5,5),(4,19),(1,1),(2,3),(0,2),(6,8),(0,1),\ldots$

Periodic expansions

Open problem

In 1D: Theorem: The continued fraction of $\alpha \in \mathbb{R}$ is periodic $\iff$ α is quadratic.

In 2D: Which $(1, \alpha, \beta)$ have a periodic JP expansion?

Periodic expansions in cubic fields

Theorem [Dubois and Paysant-Le Roux 1975]

Let $\mathbb{K}$ be a cubic real extension of $\mathbb{Q}$.
There exist $\alpha, \beta \in \mathbb{K}$ such that:

1. $\mathbb{K} = \mathbb{Q}(\alpha, \beta)$.
2. The JP expansion of $(1, \alpha, \beta)$ is **periodic**.

1 bis. Find σ.

- **Matrix formulation of JP**

Jacobi-Perron matrices

Classical formulation of JP

$$\mathbf{v} = (a, b, c) \quad \overset{\mathsf{JP}}{\longmapsto} \quad \mathbf{v}_1 = (b - \lfloor b/a \rfloor a,\ c - \lfloor c/a \rfloor a,\ a)$$

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Formulation with matrices

$\mathbf{v} = \mathbf{M}_{B,C}\mathbf{v}_1,$ where

$$\mathbf{M}_{B,C} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & B \\ 0 & 1 & C \end{pmatrix} \qquad B = \lfloor b/a \rfloor \qquad C = \lfloor c/a \rfloor$$

Jacobi-Perron matrices

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- $\mathbf{v} = \mathbf{M}_{B_1,C_1} \cdots \mathbf{M}_{B_n,C_n}\mathbf{v}_n$.
- $\mathbf{M}_{B_1,C_1} \cdots \mathbf{M}_{B_n,C_n}\mathbf{u}$ converges to $\mathbf{v}$ for all $\mathbf{u}$.

Jacobi-Perron substitutions

We choose $\sigma_{B,C} : \begin{cases} 1 & \mapsto & 3 \\ 2 & \mapsto & 13^B \\ 3 & \mapsto & 23^C \end{cases}$

- The incidence matrix of $\sigma_{B,C}$ is ${}^t\mathbf{M}_{B,C}$.
- $\sigma_{B,C}$ is unimodular Pisot irreducible.

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- Let $\sigma = \sigma_{B_1,C_1} \cdots \sigma_{B_\ell,C_\ell}$.
- $(1, \alpha, \beta)$ is the eigenvector of ${}^{\mathrm{t}}\mathbf{M}_{B,C}$ associated with the largest eigenvalue, so we are done:

The toral translation associated with (X_σ, S) is $(\mathbb{T}^2, T_{\alpha,\beta})$.

2. Prove $(\mathbb{T}^2, T_{\alpha,\beta}) \;\cong\; (X_\sigma, S)$.

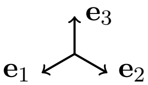
Combinatorics:

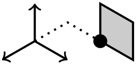
- **discrete surfaces**
- **multidimensional substitutions**
- **a definition of Rauzy fractals**

Unit faces

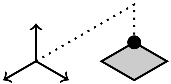
A **unit face** $[\mathbf{x}, i]^*$ consists of:

- a **position** $\mathbf{x} \in \mathbb{Z}^3$;
- a **type** $i \in \{1, 2, 3\}$.





$[(-1, 1, 0), 1]^*$



$[(-3, 0, -1), 3]^*$

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A **unit face** $[\mathbf{x}, i]^*$ consists of:

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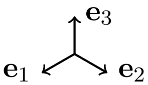

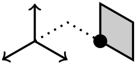

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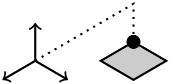


$[(-3, 0, -1), 3]^*$

$$[\mathbf{x}, 1]^* = \{\mathbf{x} + \lambda \mathbf{e}_2 + \mu \mathbf{e}_3 : \lambda, \mu \in [0, 1]\} =$$

$$[\mathbf{x}, 2]^* = \{\mathbf{x} + \lambda \mathbf{e}_1 + \mu \mathbf{e}_3 : \lambda, \mu \in [0, 1]\} =$$

$$[\mathbf{x}, 3]^* = \{\mathbf{x} + \lambda \mathbf{e}_1 + \mu \mathbf{e}_2 : \lambda, \mu \in [0, 1]\} =$$

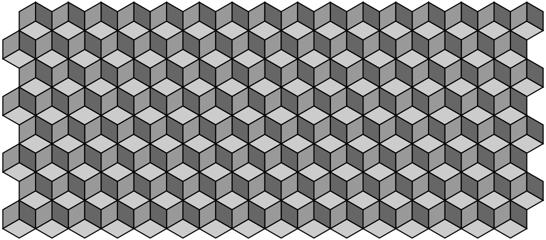
Discrete planes

Let $\mathbf{v} \in \mathbb{R}^3_{>0}$. The **discrete plane** $\Gamma_\mathbf{v}$ of normal vector $\mathbf{v}$ is

the discrete surface that "intersects" the plane $\mathcal{P}_\mathbf{v}$.

(Formally: $\Gamma_\mathbf{v} = \{[\mathbf{x}, i]^* : 0 \leqslant \langle \mathbf{x}, \mathbf{v} \rangle < \langle \mathbf{e}_i, \mathbf{v} \rangle\}$.)

$\Gamma_{(1,1,1)}$



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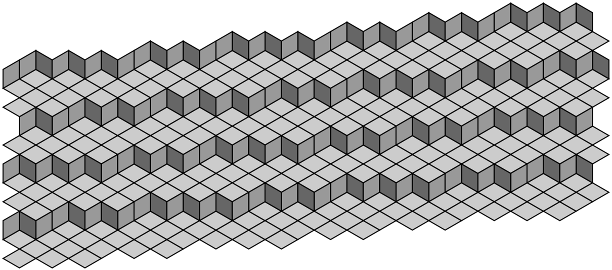
$\Gamma_{(1, \sqrt{2}, \sqrt{17})}$



Dual substitutions $\mathbf{E}_1^*(\sigma)$

Definition [Arnoux-Ito 2001]

Let $\sigma : \{1,2,3\}^* \to \{1,2,3\}^*$ such that $\det(\mathbf{M}_\sigma) = \pm 1$.

$$\mathbf{E}_1^*(\sigma)([\mathbf{x}, i]^*) \;=\; {\mathbf{M}_\sigma}^{-1}\mathbf{x} \;+\; \bigcup_{k=1,2,3} \;\bigcup_{s|\sigma(k)=pis} [\ell(s), k]^*,$$

where $\ell : \{1,2,3\}^* \to \mathbb{Z}_+^3$, $w \mapsto (|w|_1, |w|_2, |w|_3)$.

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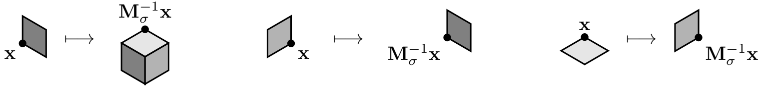
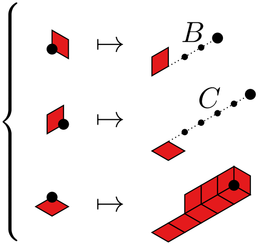
Example: $\mathbf{E}_1^*(\sigma)$ for $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$

$$\begin{array}{lcl} [\mathbf{x},1]^* & \mapsto & \mathbf{M}_\sigma{}^{-1}\mathbf{x} + [(1,0,-1),1]^* \cup [(0,1,-1),2]^* \cup [(0,0,0),3]^* \\ [\mathbf{x},2]^* & \mapsto & \mathbf{M}_\sigma{}^{-1}\mathbf{x} + [(0,0,0),1]^* \\ [\mathbf{x},3]^* & \mapsto & \mathbf{M}_\sigma{}^{-1}\mathbf{x} + [(0,0,0),2]^* \end{array}$$



Jacobi-Perron $\mathbf{E}_1^*$ substitutions

$$\begin{cases} [\mathbf{0},1]^* & \mapsto \quad [(B,0,0),2]^* \\ [\mathbf{0},2]^* & \mapsto \quad [(C,0,0),3]^* \\ [\mathbf{0},3]^* & \mapsto \quad [\mathbf{0},1]^* \cup \bigcup_{k=0}^{B-1} [(k,0,0),2]^* \cup \bigcup_{\ell=0}^{C-1} [(\ell,0,0),3]^* \end{cases}$$



(here: $B = 3$, $C = 5$)

$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

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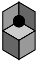
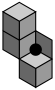
$\mathbf{E}_1^*(\sigma)($  $)$



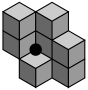
$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)^2($  $)$



$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)^3($$)$



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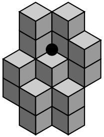
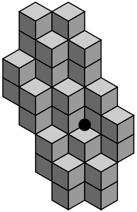


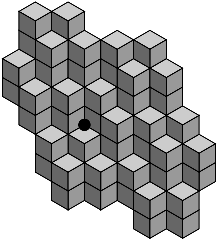
$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)^5($$)$



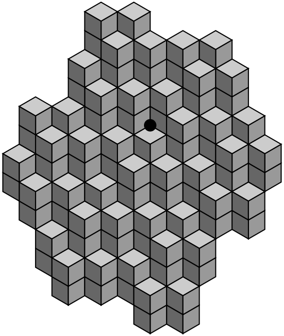
$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)^6($  $)$



$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)^7($$)$



$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112)$

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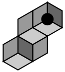
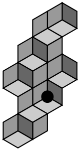
$\mathbf{E}_1^*(\sigma)($  $)$



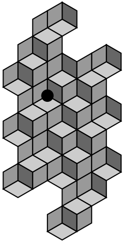
$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112)$

$\mathbf{E}_1^*(\sigma)^2($$)$



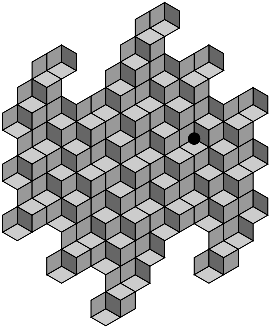
$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112)$

$\mathbf{E}_1^*(\sigma)^3($$)$



$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112)$

$\mathbf{E}_1^*(\sigma)^4($  $)$



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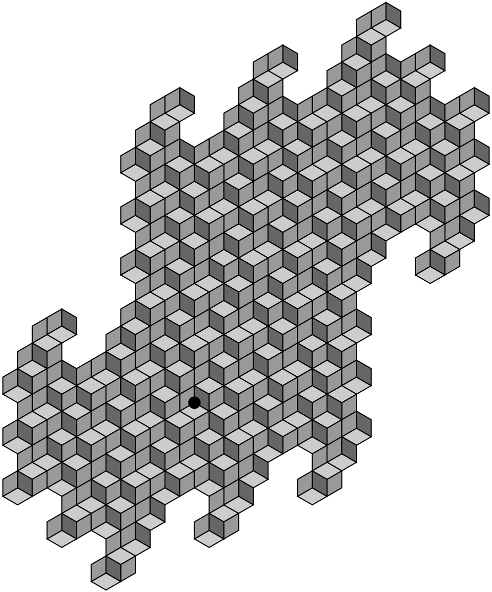
$\mathbf{E}_1^*(\sigma)^5($  $)$



$$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1)$$

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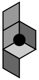
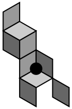
$\mathbf{E}_1^*(\sigma)($  $)$



$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)^2($$)$



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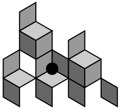

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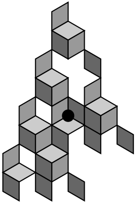

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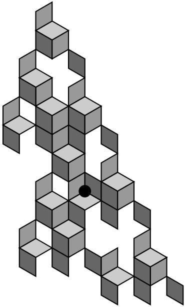
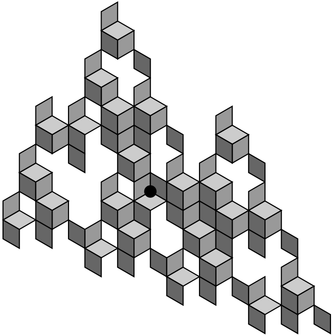


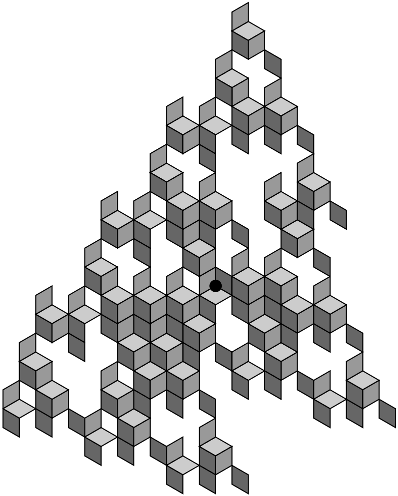
$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)^6($  $)$



$\mathbf{E}_1^*(\sigma)(1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1)$

$\mathbf{E}_1^*(\sigma)^7($  $)$



$\mathbf{E}_1^*(\sigma)$ and discrete planes

Theorem [Arnoux-Ito 2001, Fernique 2007]

$\mathbf{E}_1^*(\sigma)^n($$)$ $\subseteq$ **a discrete plane**.

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Even better: $\mathbf{E}_1^*(\sigma)(\Gamma_{\mathbf{v}}) = \Gamma_{{}^{t}\mathbf{M}_\sigma \mathbf{v}}$.

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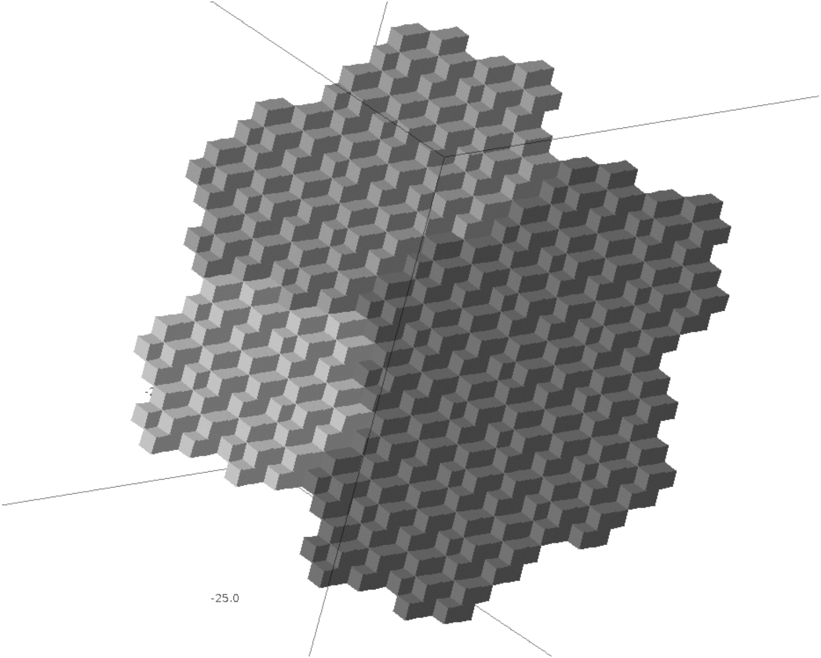
Theorem [Arnoux-Ito 2001]

$[\mathbf{x}, i]^* \neq [\mathbf{y}, j]^* \in \Gamma_{\mathbf{v}}$

$\implies \mathbf{E}_1^*(\sigma)([\mathbf{x}, i]^*)$ and $\mathbf{E}_1^*(\sigma)([\mathbf{y}, j]^*)$ are disjoint.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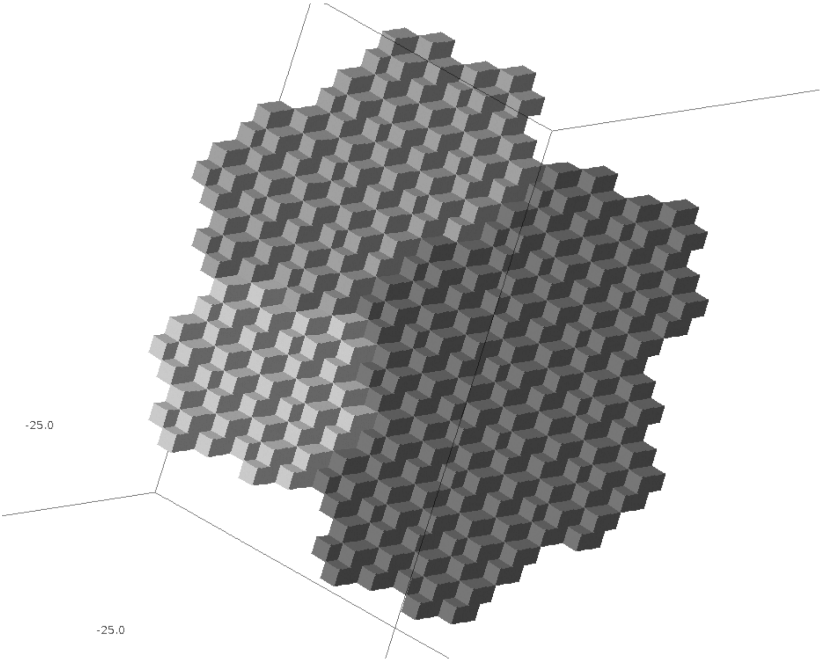

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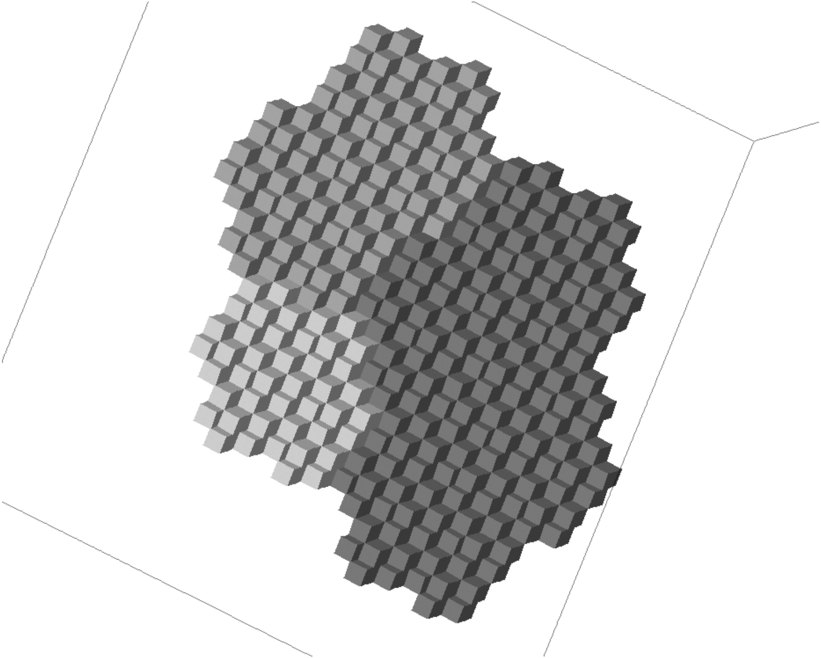

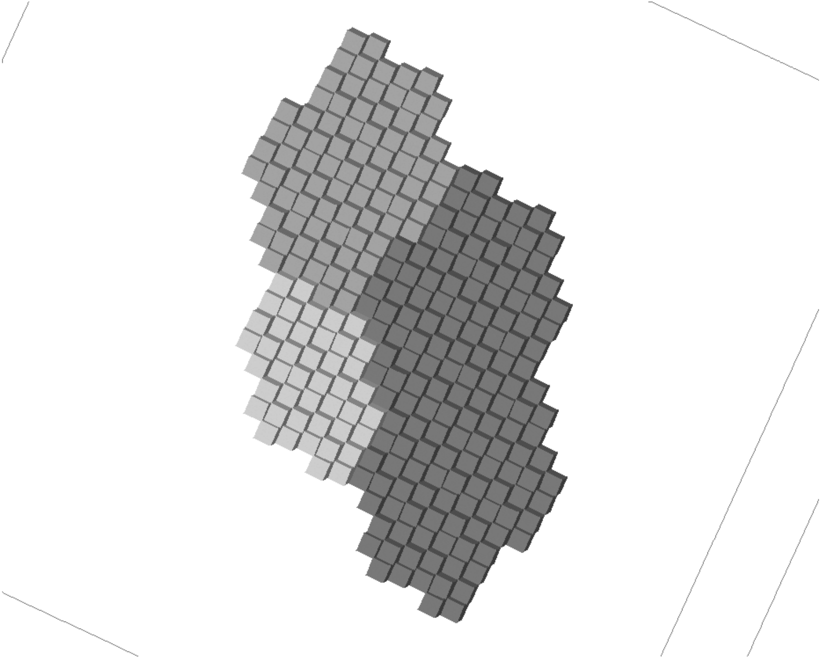

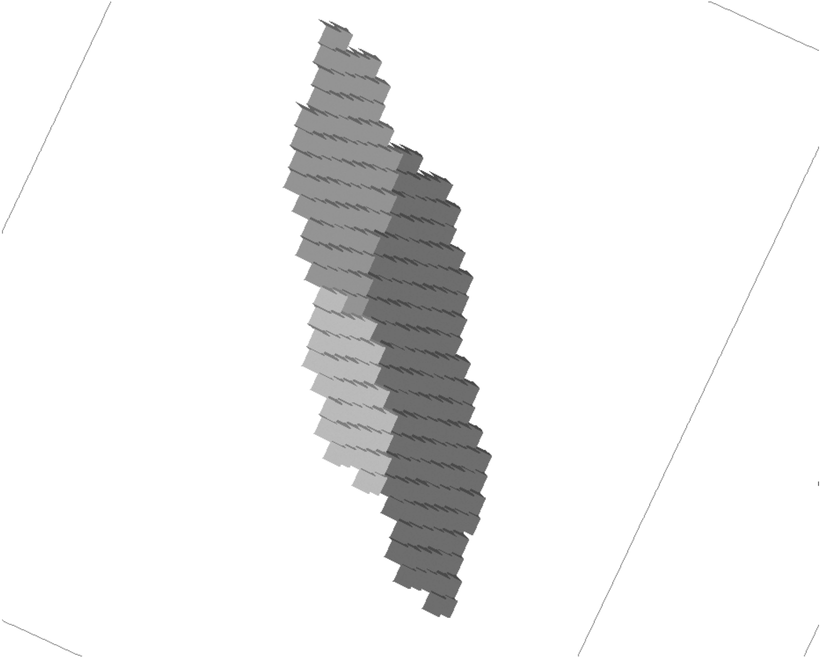

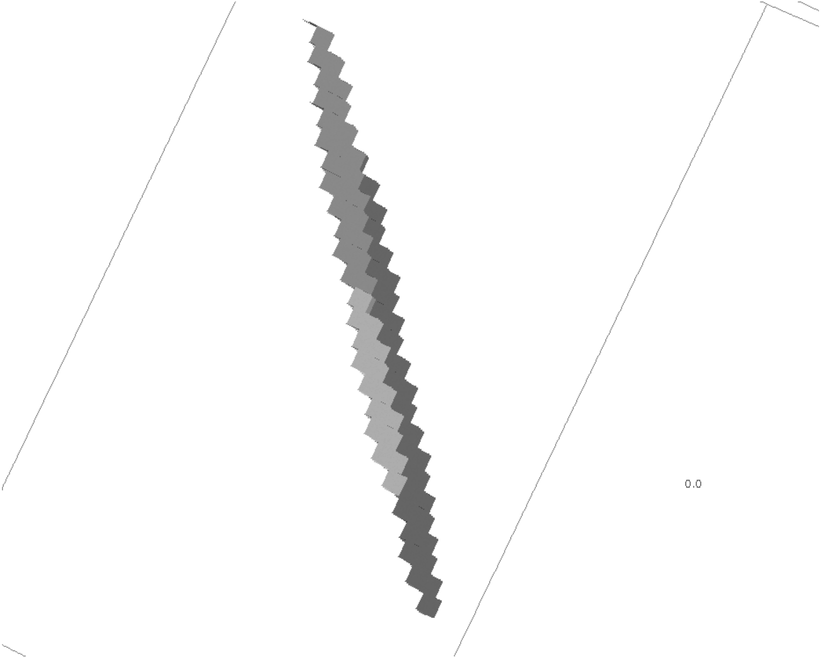
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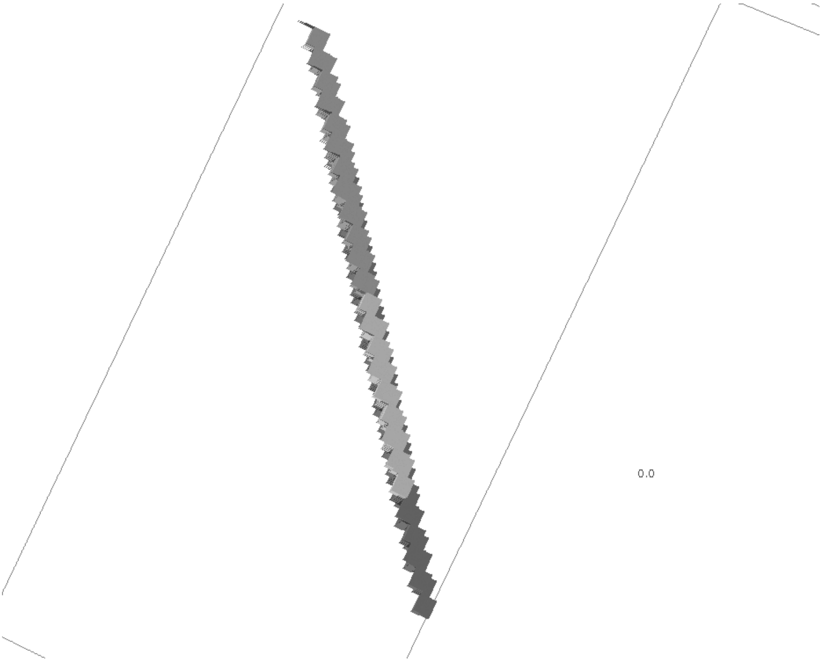
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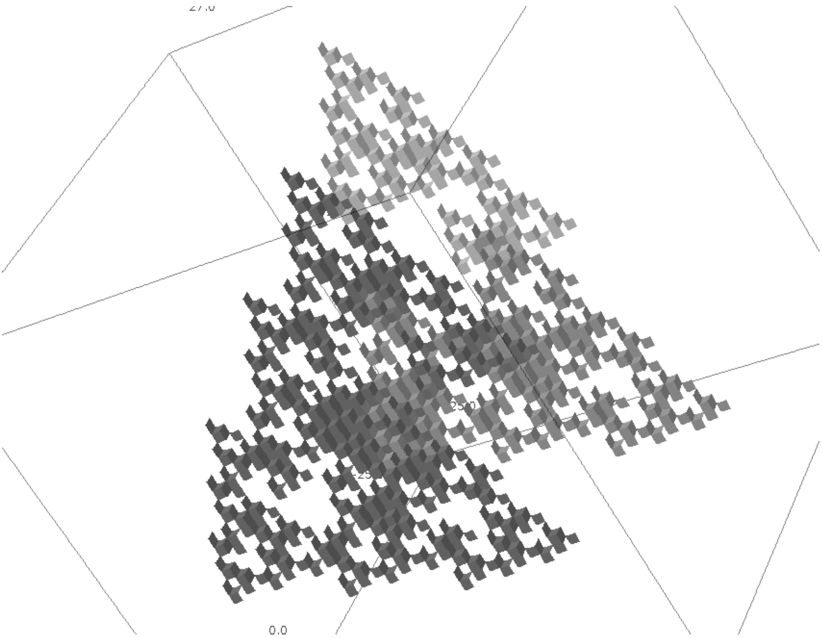

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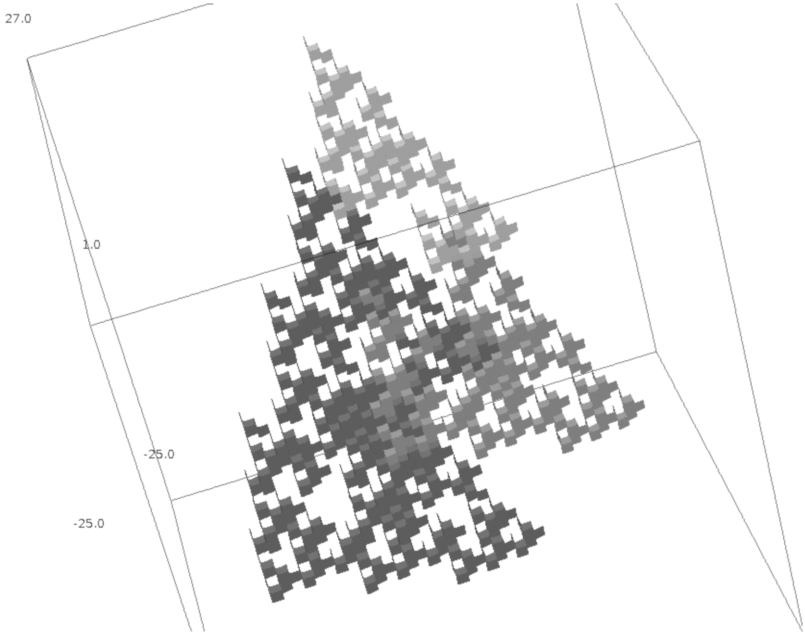
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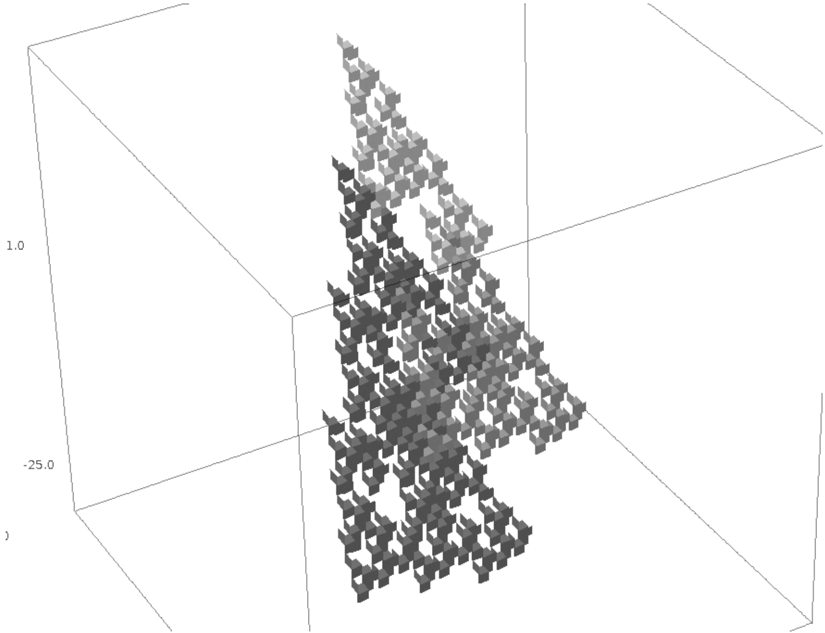
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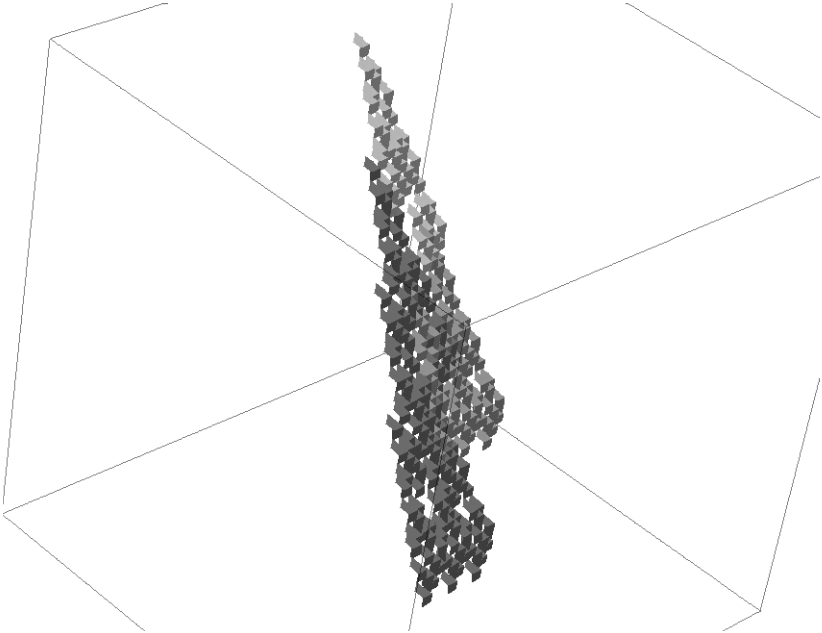
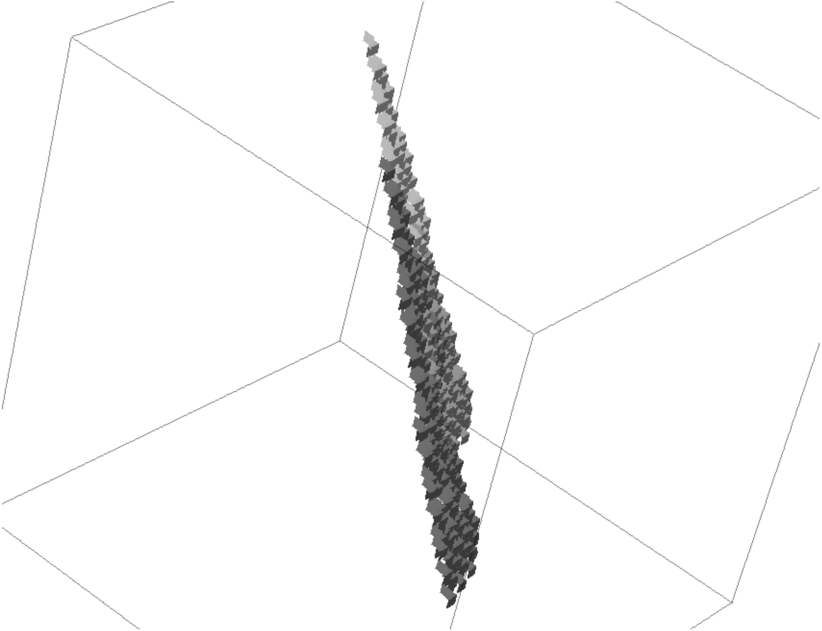
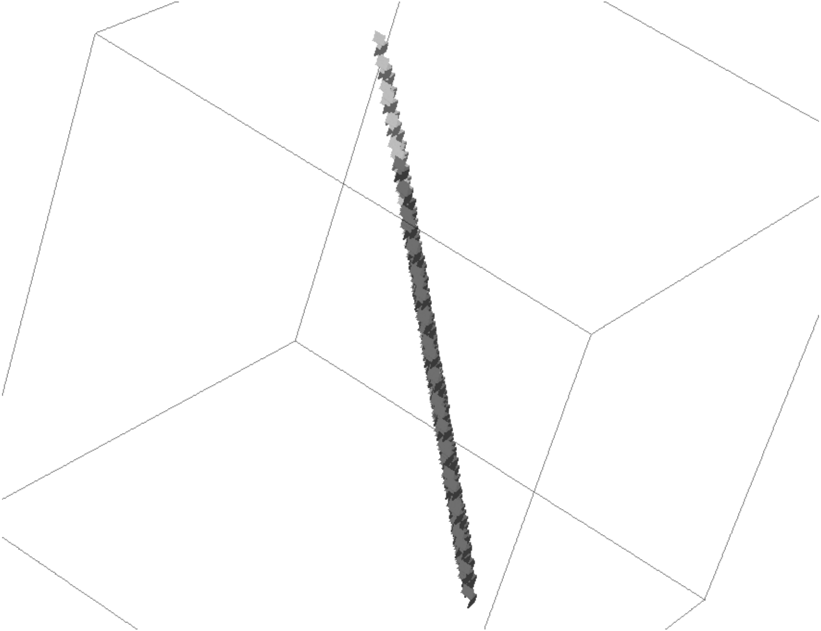






Definition of Rauzy fractals using $\mathbf{E}_1^*(\sigma)$

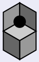
Idea: **renormalize** $\mathbf{E}_1^*(\sigma)^n(\mathcal{U})$ with $n \to \infty$.

$\mathcal{U}$

Definition of Rauzy fractals using $\mathbf{E}_1^*(\sigma)$

Idea: renormalize $\mathbf{E}_1^*(\sigma)^n(\mathcal{U})$ with $n \to \infty$.

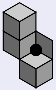
$\mathbf{E}_1^*(\mathcal{U})$



Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

Idea: renormalize $\mathbf{E}_1^*(\sigma)^n(\mathcal{U})$ with $n \to \infty$.

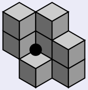
$\mathbf{E}_1^{*2}(\mathcal{U})$



Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

Idea: renormalize $\mathbf{E}_1^*(\sigma)^n(\mathcal{U})$ with $n \to \infty$.

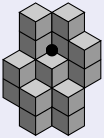
$\mathbf{E}_1^{*3}(\mathcal{U})$



Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

Idea: renormalize $\mathbf{E}_1^*(\sigma)^n(\mathcal{U})$ with $n \to \infty$.

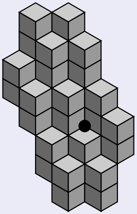
$\mathbf{E}_1^{*4}(\mathcal{U})$



Definition of Rauzy fractals using $\mathbf{E}_1^*(\sigma)$

Idea: renormalize $\mathbf{E}_1^*(\sigma)^n(\mathcal{U})$ with $n \to \infty$.

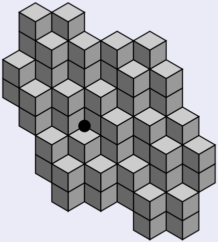
$\mathbf{E}_1^{*5}(\mathcal{U})$



Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

Idea: renormalize $\mathbf{E}_1^*(\sigma)^n(\mathcal{U})$ with $n \to \infty$.

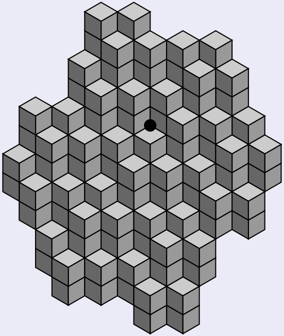
$\mathbf{E}_1^{*6}(\mathcal{U})$



Definition of Rauzy fractals using $\mathbf{E}_1^*(\sigma)$

Idea: renormalize $\mathbf{E}_1^*(\sigma)^n(\mathcal{U})$ with $n \to \infty$.

$\mathbf{E}_1^{*7}(\mathcal{U})$



Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

σ is Pisot:

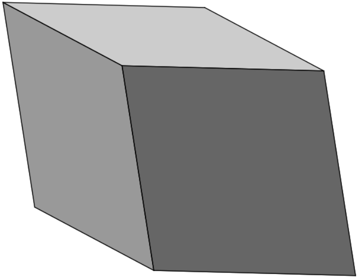
- One expanding direction of $\mathbf{M}_\sigma$ (Pisot eigenvalue $|\beta| > 1$).
- Two contracting directions of $\mathbf{M}_\sigma$ (eigenvalues $|\beta'|, |\beta''| < 1.$).
- Let $\mathbb{P}_\mathsf{c}$ be the contracting plane of $\mathbf{M}_\sigma$ spanned by $\mathbf{v}_{\beta'}, \mathbf{v}_{\beta''}$.
- Let $\pi : \mathbb{R}^3 \to \mathbb{P}_\mathsf{c}$ be the projection on $\mathbb{P}_\mathsf{c}$ along $\mathbf{v}_\beta$.
- **So:**

Renormalization $= \mathrm{M}_\sigma \circ \pi$

Definition of Rauzy fractals using $\mathbf{E}_1^*(\sigma)$

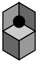
$\mathcal{U}$

$\pi(\mathcal{U})$

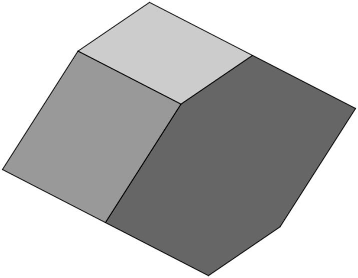


Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

$\mathbf{E}_1^*(\sigma)(\mathcal{U})$

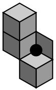


$\mathbf{M}_\sigma \pi(\mathbf{E}_1^*(\sigma)(\mathcal{U}))$

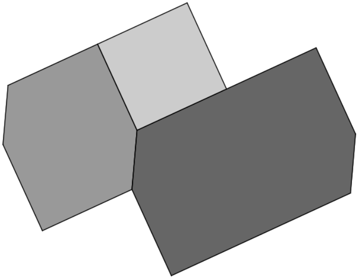


Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

$\mathbf{E}_1^*(\sigma)^2(\mathcal{U})$

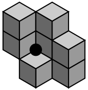


$\mathbf{M}_\sigma{}^2\pi(\mathbf{E}_1^*(\sigma)^2(\mathcal{U}))$

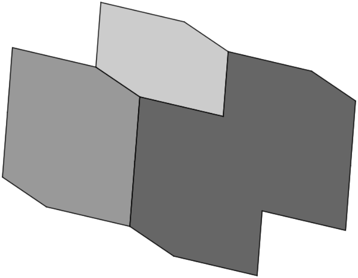


Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

$\mathbf{E}_1^*(\sigma)^3(\mathcal{U})$

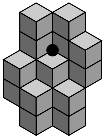


$\mathbf{M}_\sigma{}^3\pi(\mathbf{E}_1^*(\sigma)^3(\mathcal{U}))$

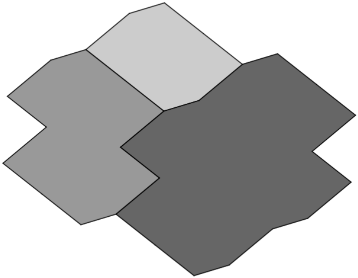


Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

$\mathbf{E}_1^*(\sigma)^4(\mathcal{U})$

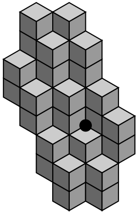


$\mathbf{M}_\sigma{}^4\pi(\mathbf{E}_1^*(\sigma)^4(\mathcal{U}))$

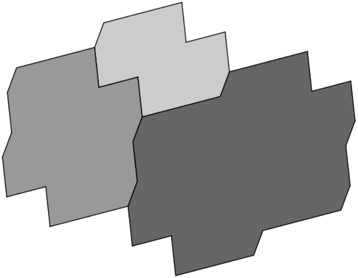


Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

$\mathbf{E}_1^*(\sigma)^5(\mathcal{U})$

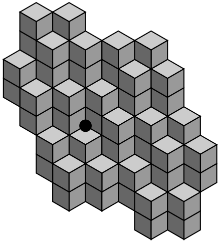


$\mathbf{M}_{\sigma}{}^5\pi(\mathbf{E}_1^*(\sigma)^5(\mathcal{U}))$

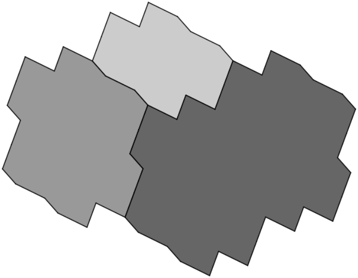


Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

$\mathbf{E}_1^*(\sigma)^6(\mathcal{U})$

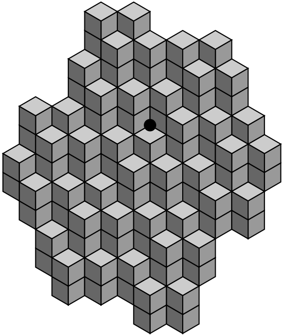


$\mathbf{M}_\sigma{}^6\pi(\mathbf{E}_1^*(\sigma)^6(\mathcal{U}))$

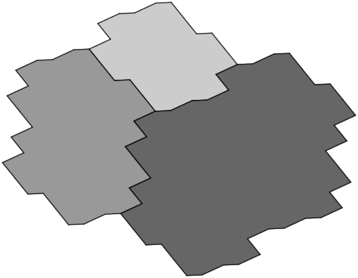


Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

$\mathbf{E}_1^*(\sigma)^7(\mathcal{U})$

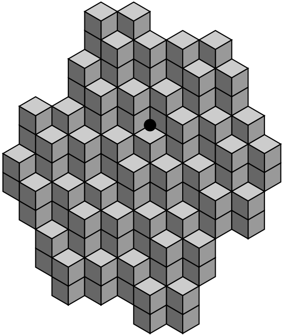


$\mathbf{M}_\sigma{}^7\pi(\mathbf{E}_1^*(\sigma)^7(\mathcal{U}))$

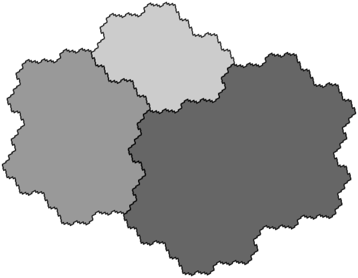


Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

$\mathbf{E}_1^*(\sigma)^7(\mathcal{U})$

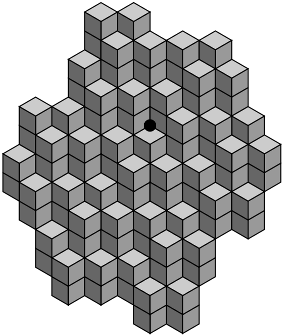


$\mathbf{M}_\sigma{}^\infty \pi(\mathbf{E}_1^*(\sigma)^\infty(\mathcal{U}))$

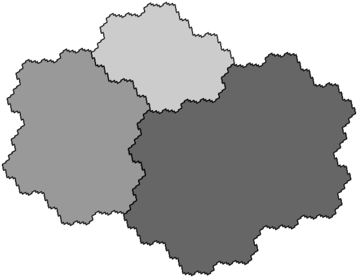


Definition of Rauzy fractals using $\mathrm{E}_1^*(\sigma)$

$\mathbf{E}_1^*(\sigma)^7(\mathcal{U})$



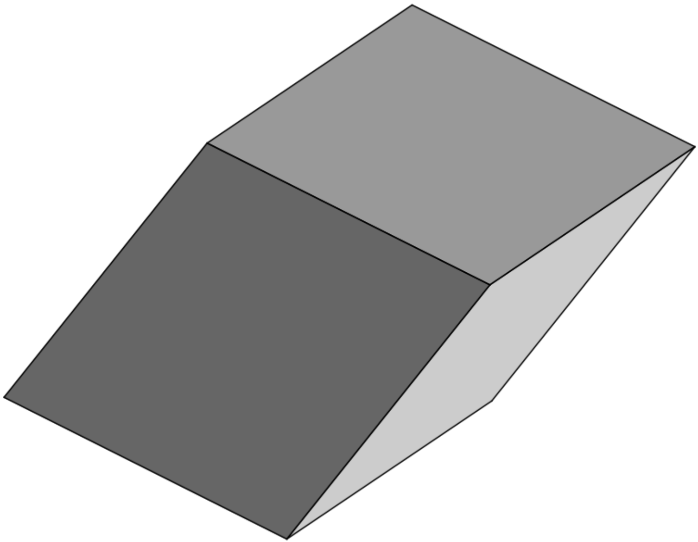
$\mathbf{M}_\sigma{}^\infty \pi(\mathbf{E}_1^*(\sigma)^\infty(\mathcal{U}))$



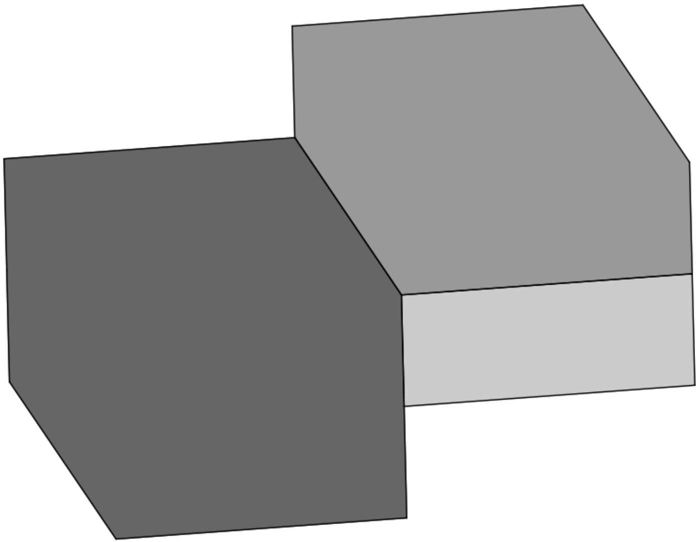
Definition [Arnoux-Ito 2001]

The **Rauzy fractal** of σ is the Hausdorf limit of $\mathbf{M}_\sigma{}^n \pi(\mathbf{E}_1^*(\sigma)^n(\mathcal{U}))$ as $n \to \infty$.

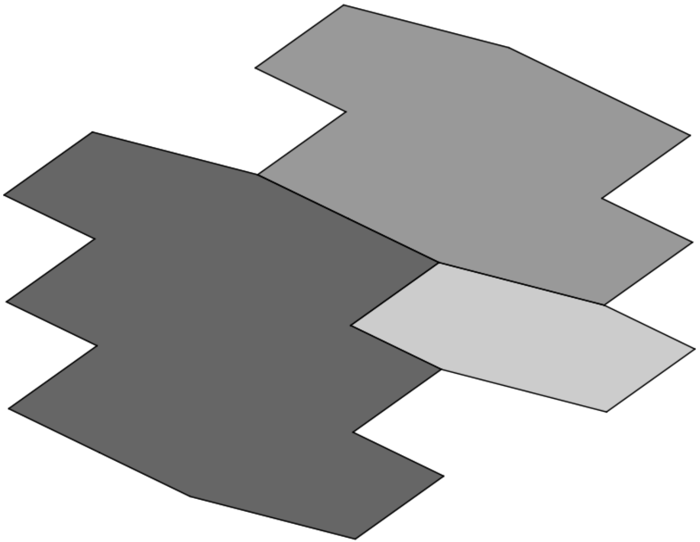
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



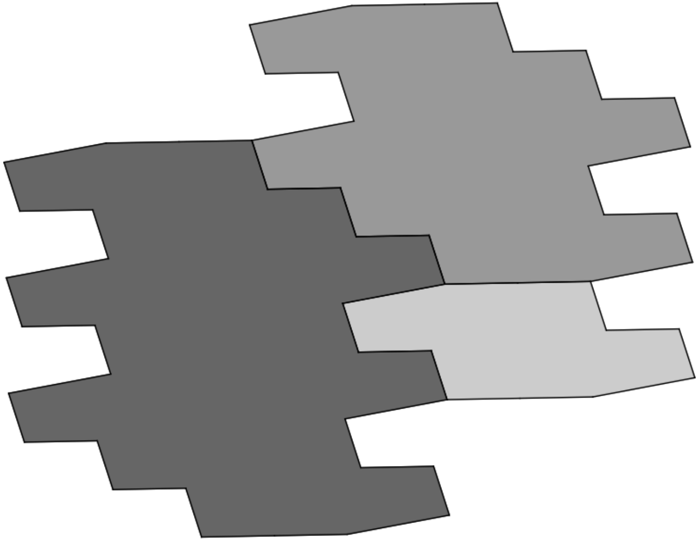
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



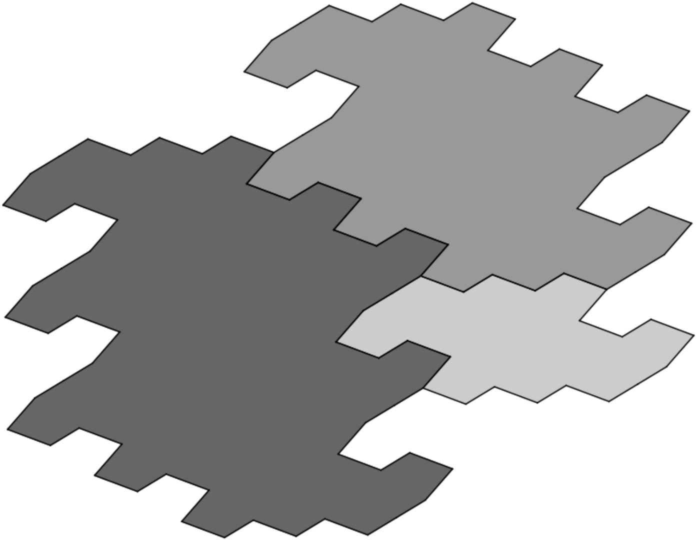
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



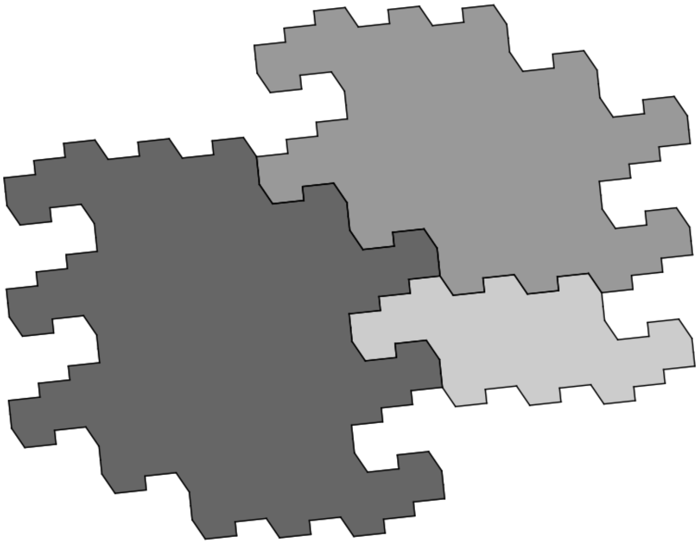
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



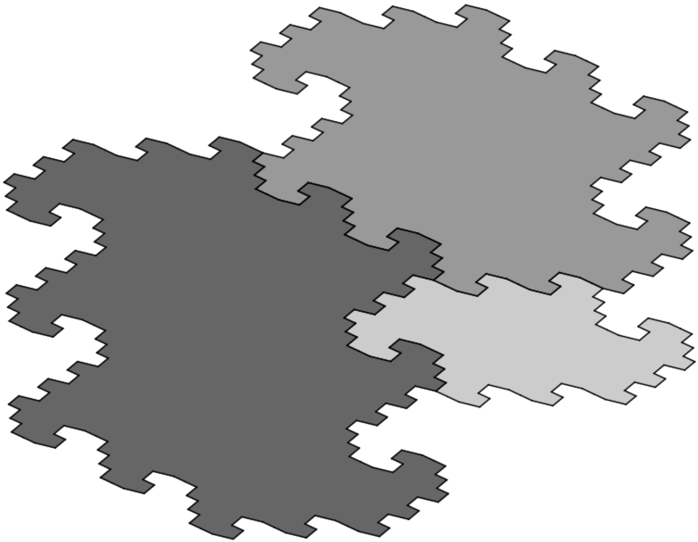
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



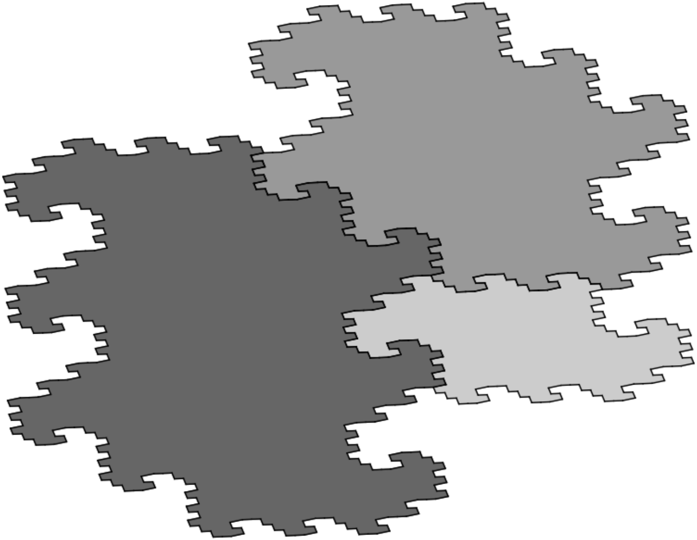
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



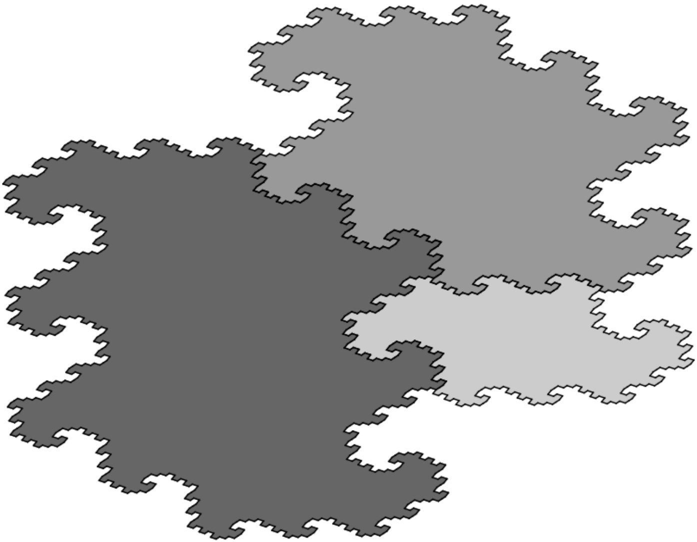
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



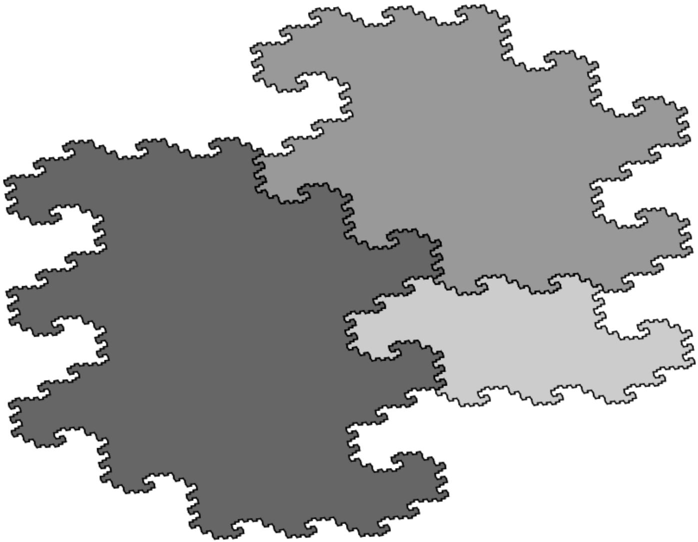
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



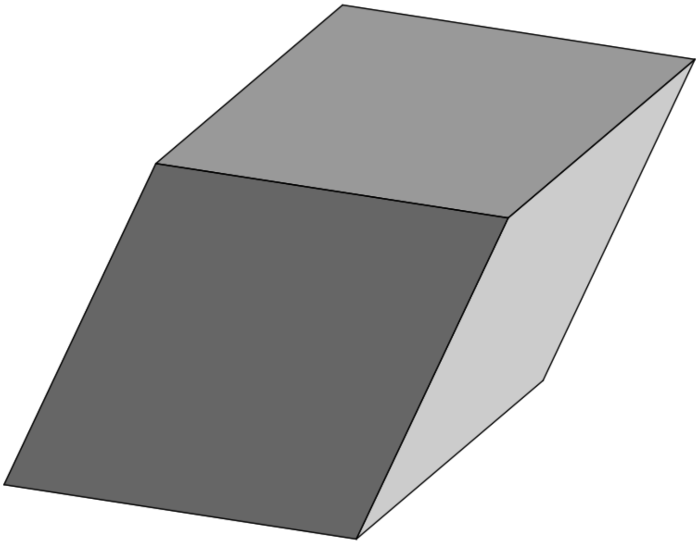
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



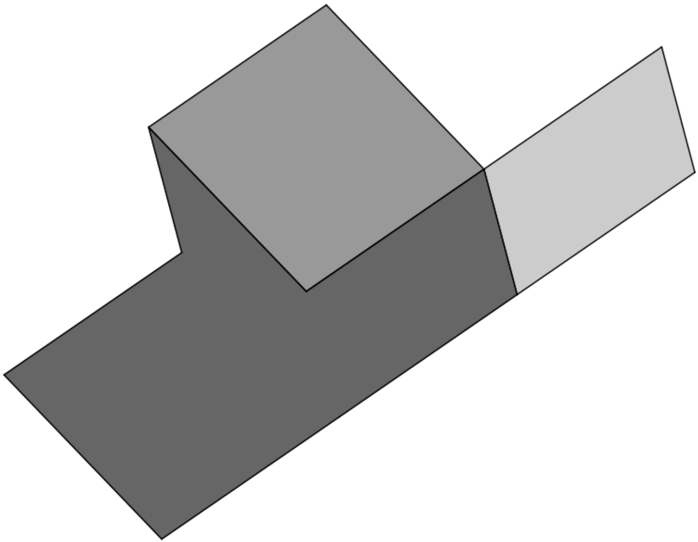
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



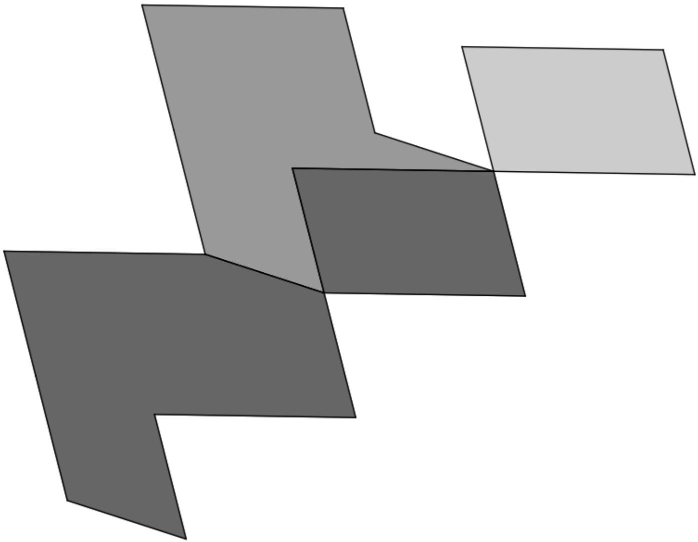
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



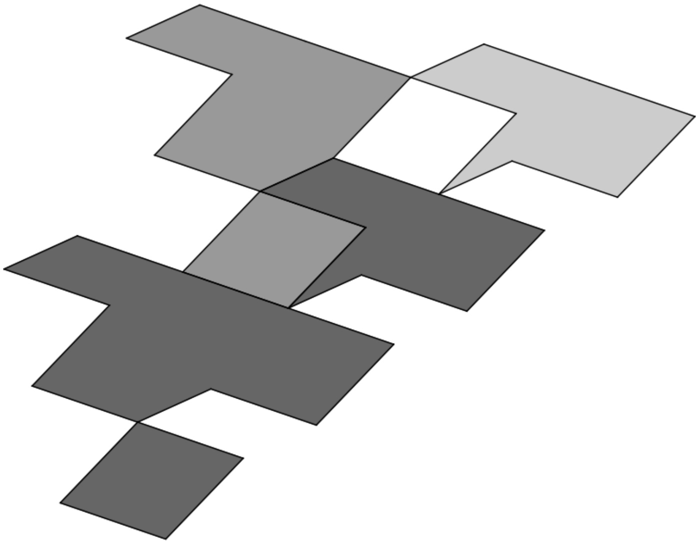
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



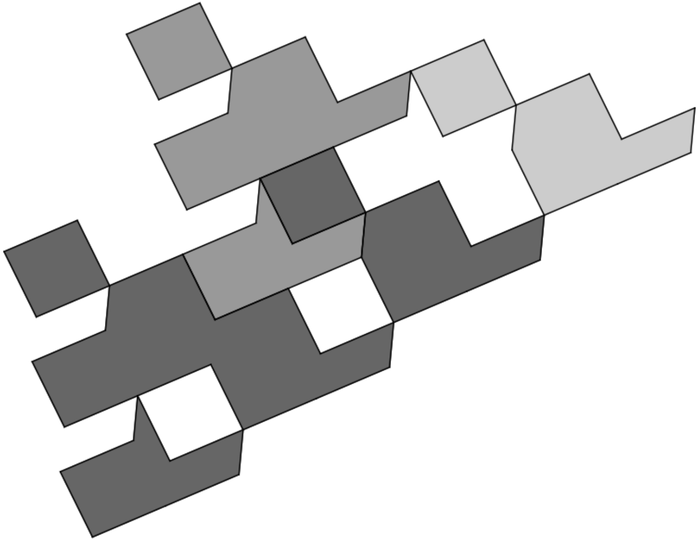
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



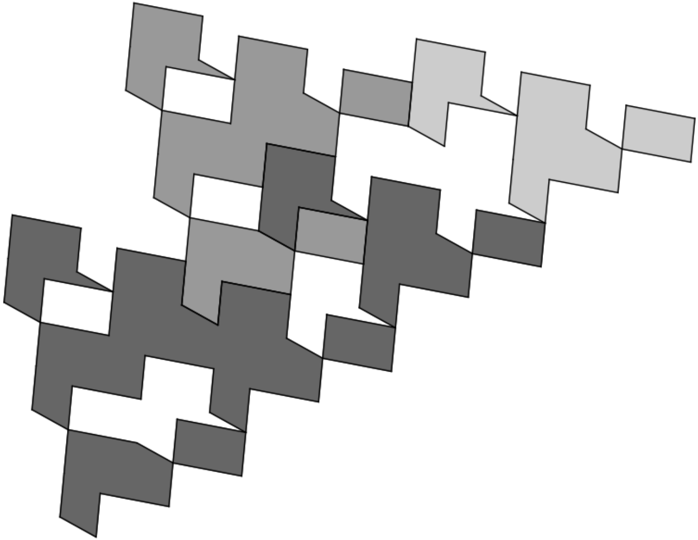
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



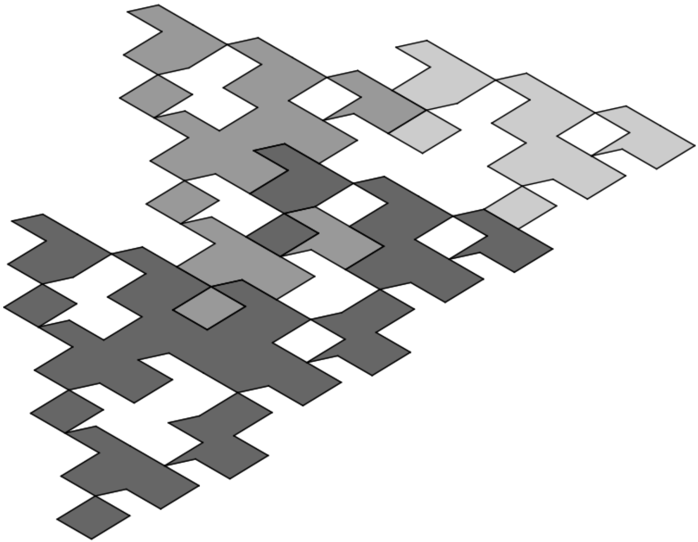
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



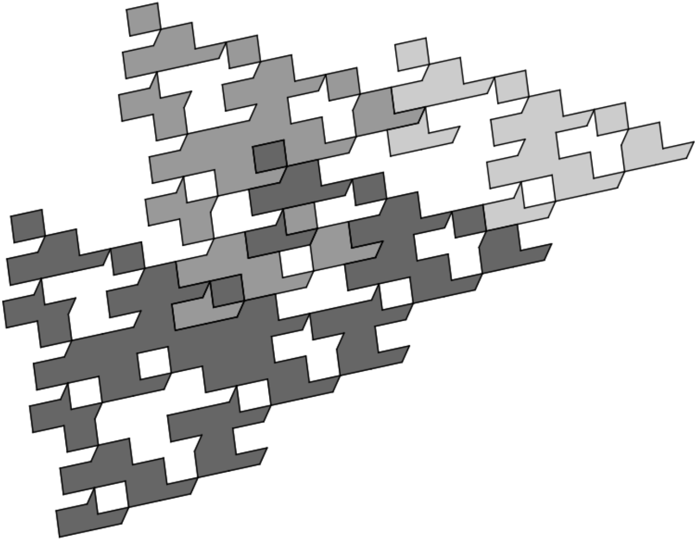
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



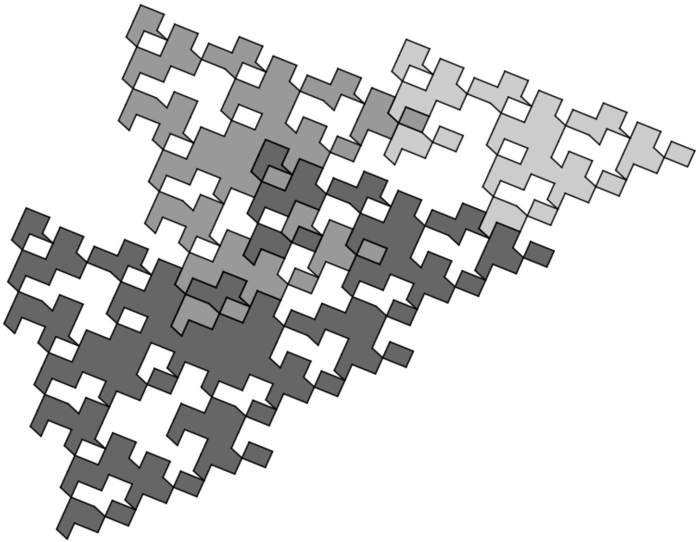
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



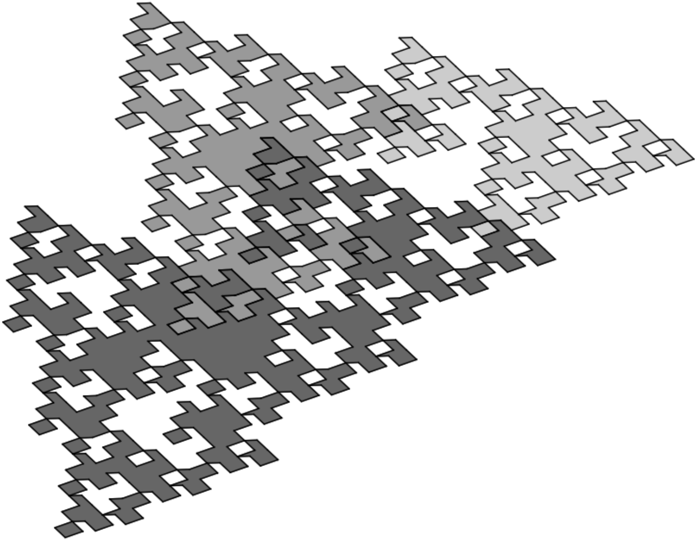
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



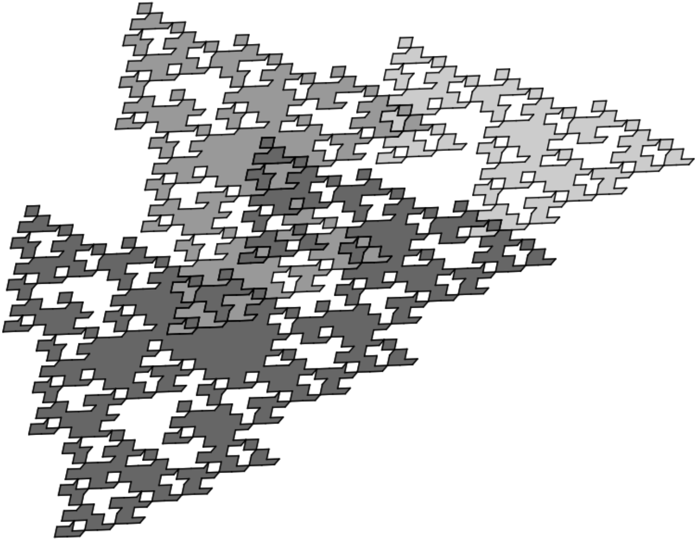
Rauzy fractal of $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



Rauzy fractal of $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



Rauzy fractal of $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



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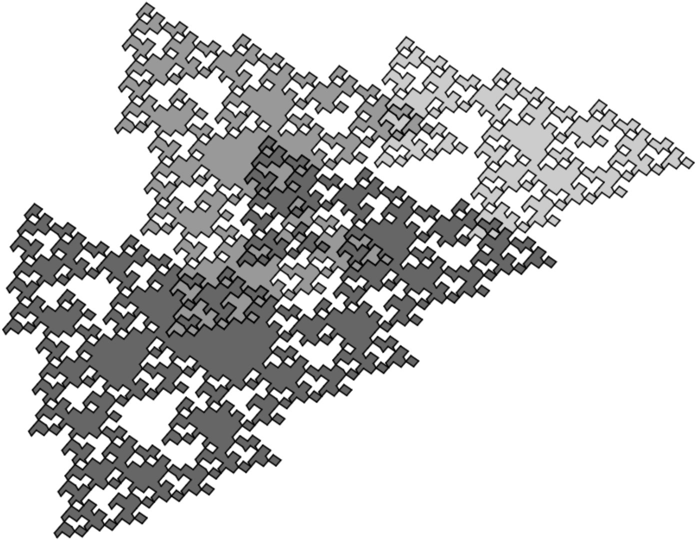

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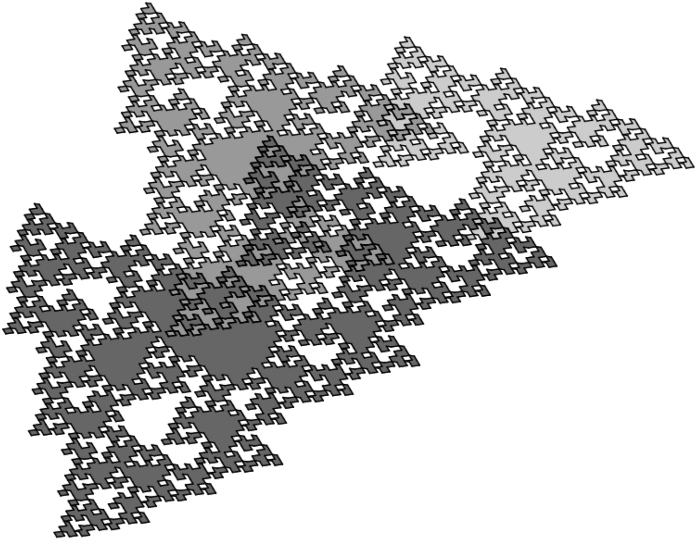

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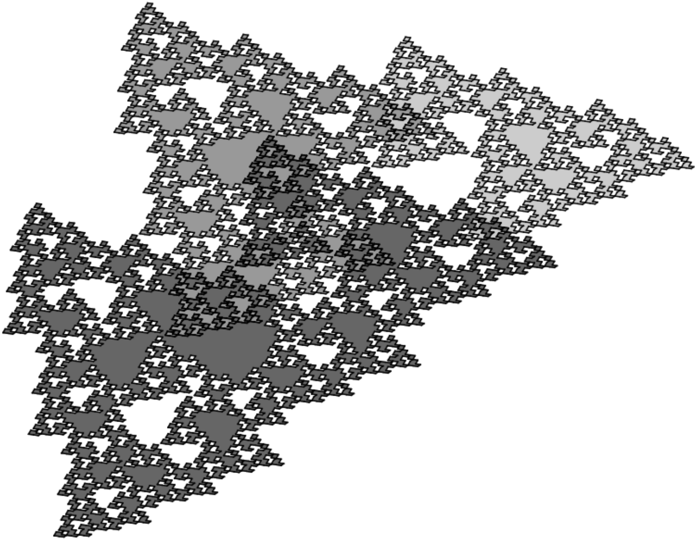

Back to our original goal:

$(\mathbb{T}^2, T_{\alpha,\beta}) \;\cong\; (X_\sigma, S)$

with $\sigma = \sigma_{B_1,C_1} \cdots \sigma_{B_\ell,C_\ell}$.

Combinatorial criterion

- Many crieteria/techniques exist for a given single σ, **but:**
- We must deal with the infinite family of all the $\sigma_{B_1,C_1} \cdots \sigma_{B_\ell,C_\ell}$ that can arise from α, β.

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Theorem [Ito-Rao 2006]

The isomorphism holds if and only if $\mathbf{E}_1^*(\sigma)^n([\mathbf{0}, i]^*)$ contains arbitrarily large balls as $n \to \infty$, for $i \in \{1, 2, 3\}$.

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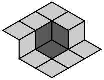
➥ How do we prove that balls grow in $\mathbf{E}_1^*(\sigma)^n([\mathbf{0}, i]^*)$?

The annulus property

Annulus A of P:

- A is edge-connected,
- A "surrounds" P.



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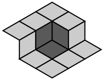

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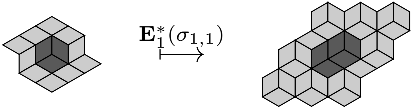

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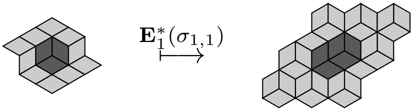

The annulus property: $\mathbf{E}_1^*(\sigma)(\text{annulus}) = \text{annulus}$.

If

1. $\mathbf{E}_1^*(\sigma)([\mathbf{0}, i]^*)^n$ contains an annulus for some n.
2. Annulus property for $\mathbf{E}_1^*(\sigma)$.

Then $\mathbf{E}_1^*(\sigma)([\mathbf{0}, i]^*)^n$ contains arbitrarily large balls. (induction)

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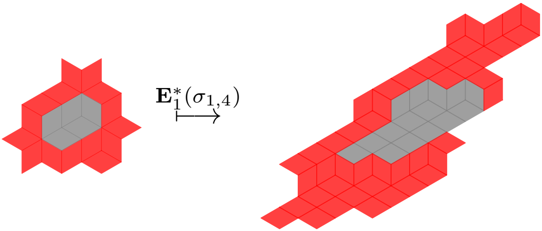

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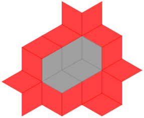
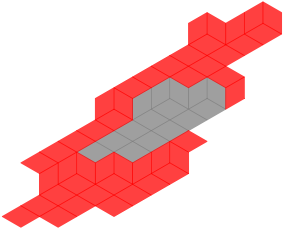


$$\mathbf{E}_1^*(\sigma_{1,4})$$



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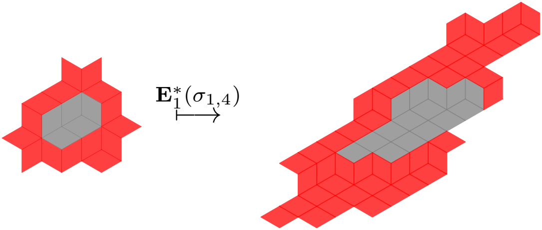
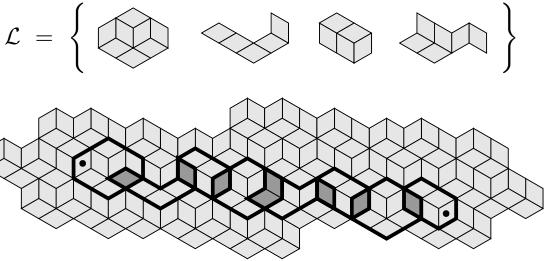


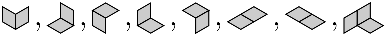
- ➥ We have to be more careful.
- ➥ Stronger assumptions on A: **covering properties**.

$\mathcal{L}$-coverings



P is **$\mathcal{L}$-covered** if $\forall f, g \in P$, there is an **$\mathcal{L}$-path from f to g**.

$\mathcal{L}$-covering and Jacobi-Perron

- We will use $\mathcal{L}_{\mathsf{JP}} = \{$  $\}$.

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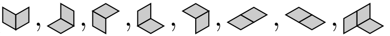
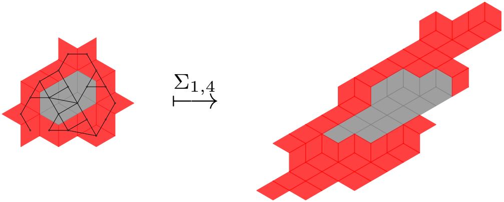
- We will use $\mathcal{L}_{\mathsf{JP}} = \{$  $\}$.
- Let's look at our problem again:



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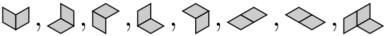
- We will use $\mathcal{L}_{\mathsf{JP}} = \{$  $\}$.
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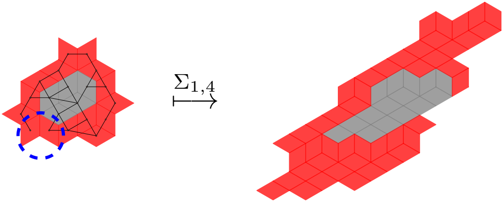

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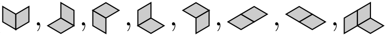
- We will use $\mathcal{L}_{\mathsf{JP}} = \{$  $\}$.
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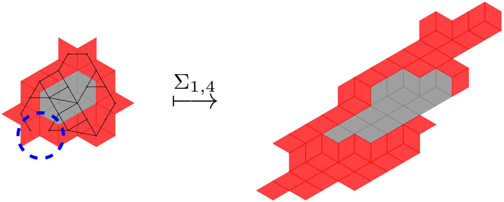

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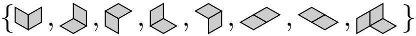
- We will use $\mathcal{L}_{\mathsf{JP}} =$ .
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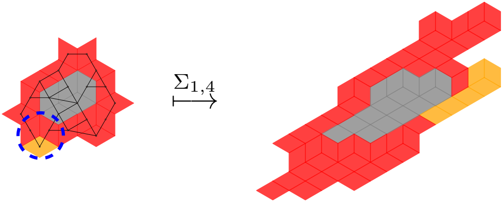

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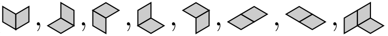
- We will use $\mathcal{L}_{\mathsf{JP}} = \{$  $\}$.
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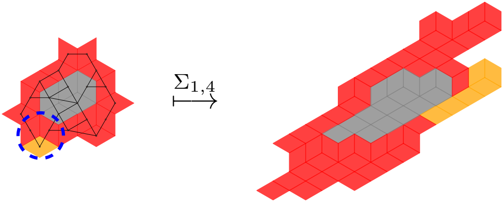

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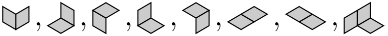
- We will use $\mathcal{L}_{\mathrm{JP}} = \{$  $\}$.
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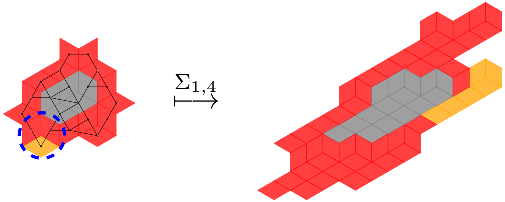

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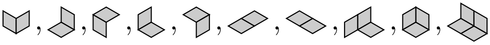
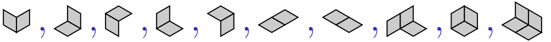
- New $\mathcal{L}_{\mathrm{JP}} = \{$  $\}$.

Why $\mathcal{L}_{\mathsf{JP}} = \{$  $\}$?

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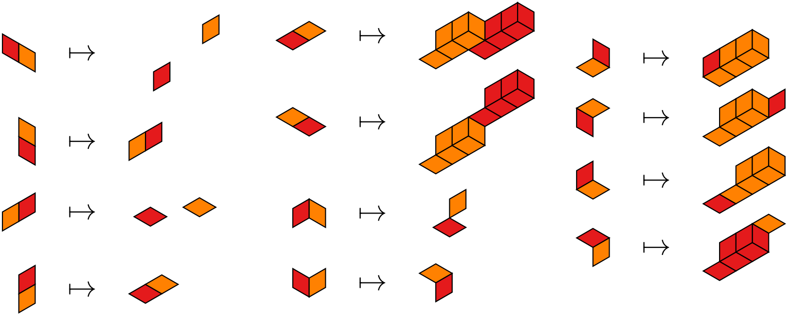
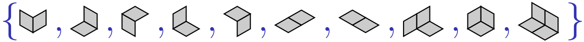


Why $\mathcal{L}_{\mathsf{JP}} =$  ?

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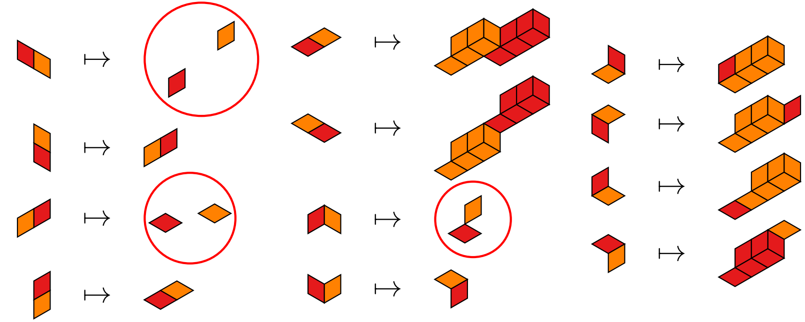
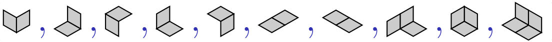


We remove the "bad" patterns. . .

Why $\mathcal{L}_{\mathsf{JP}} = \{$  $\}$?

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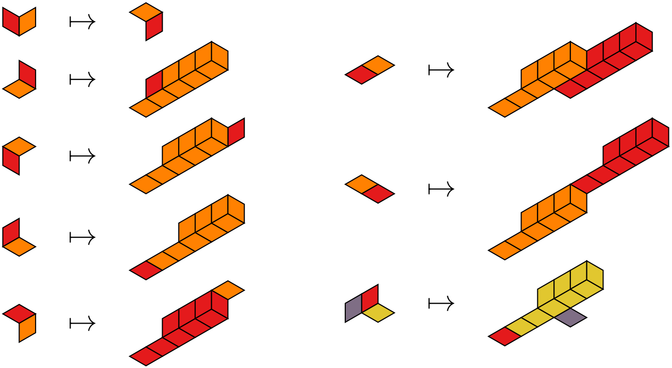
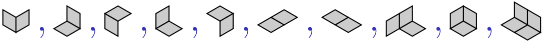


Why $\mathcal{L}_{\mathsf{JP}} = \{$  $\}$?

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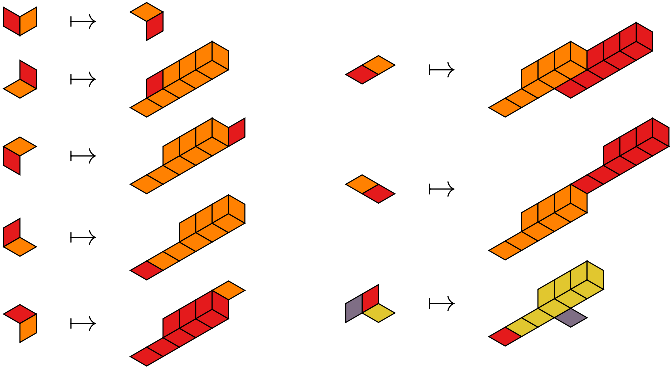

- (Plus  and  to guarantee **strong** $\mathcal{L}_{\mathsf{JP}}$-covering.)
- The images are strongly $\mathcal{L}_{\mathsf{JP}}$-covered: stability.

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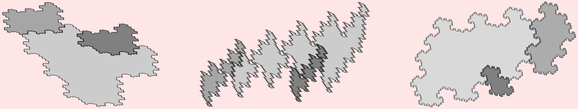
- The Rauzy fractal is connected.



Conclusion

We have reached our initial objective:

Corollary

Let $\mathbb{K}$ be a cubic real extension of $\mathbb{Q}$.

- There exist $\alpha, \beta \in \mathbb{K}$ with periodic JP expansion $(B_1, C_1), \ldots, (B_\ell, C_\ell)$ such that $\mathbb{K} = \mathbb{Q}(\alpha, \beta)$. [Dubois and Paysant-Le Roux 1975]

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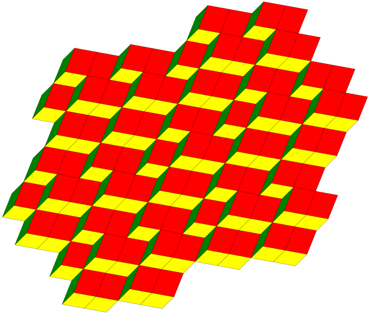
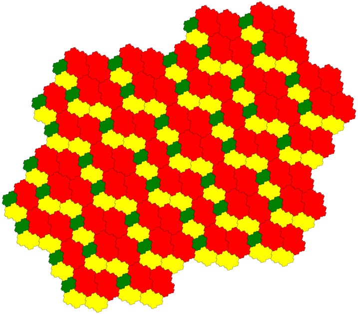
- For $\sigma = \sigma_{B_1,C_1} \cdots \sigma_{B_\ell,C_\ell}$, we have $(\mathbb{T}^2, T_{\alpha,\beta}) \cong (X_\sigma, S)$. [Geometrical combinatorial methods]

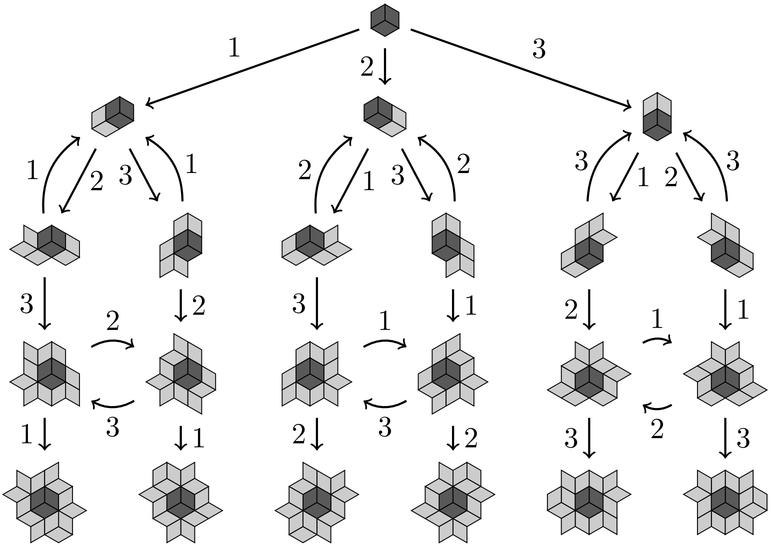
Thank you for your attention.

Any questions?

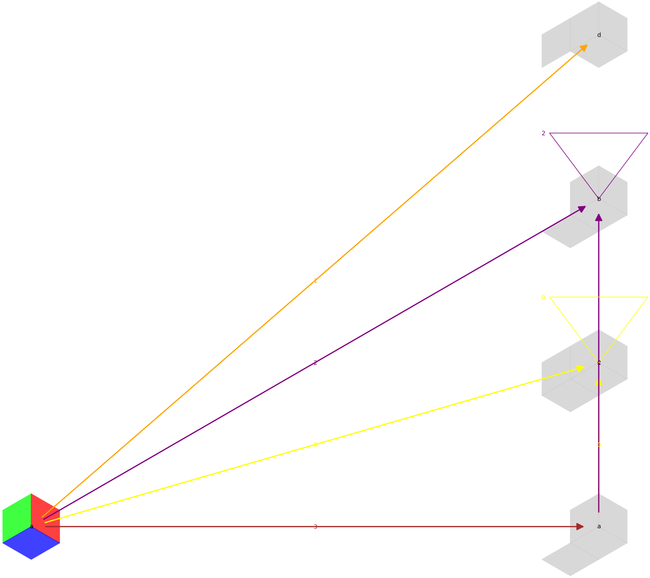




Dream case (Arnoux-Rauzy)



Nightmare (Jacobi-Perron)



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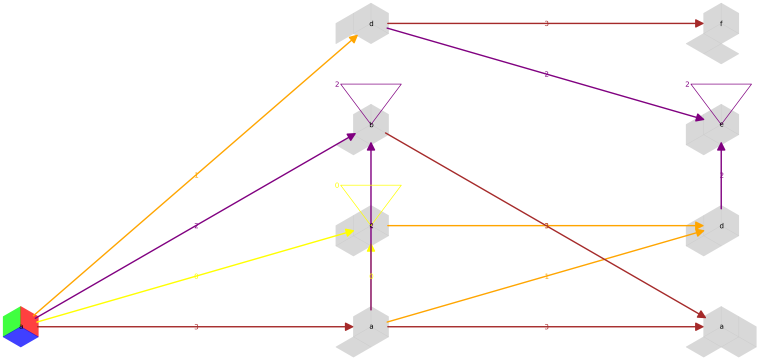

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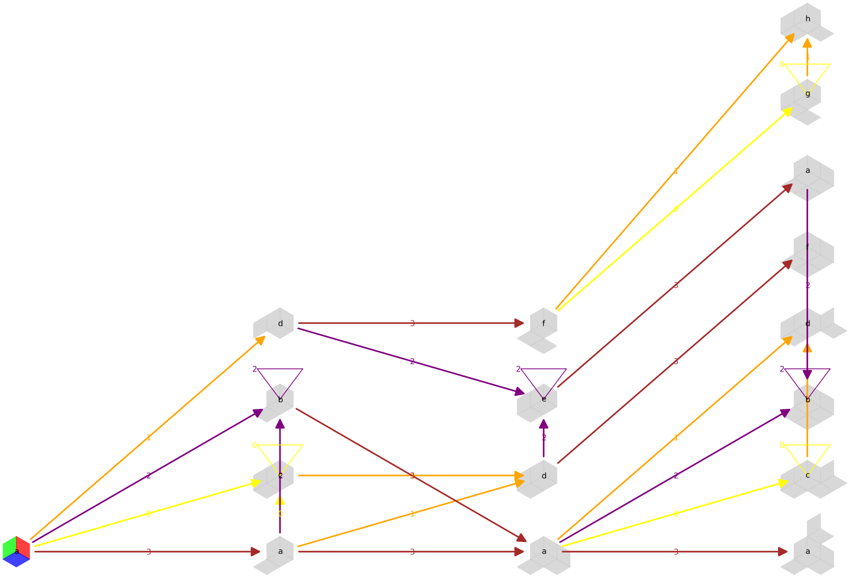

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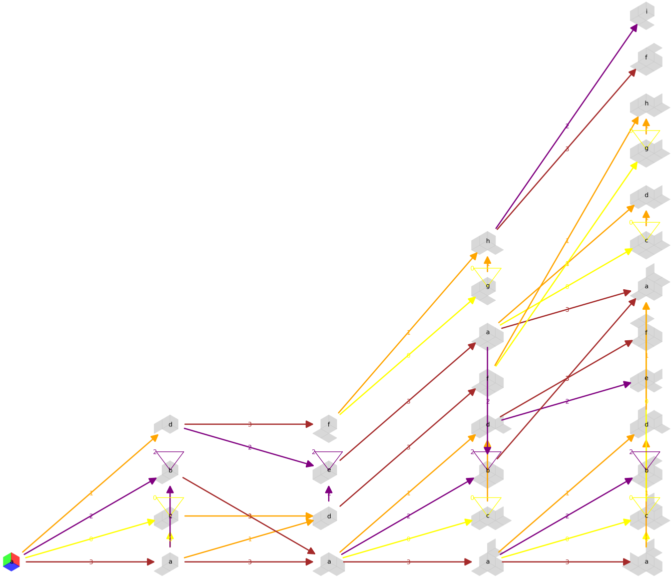

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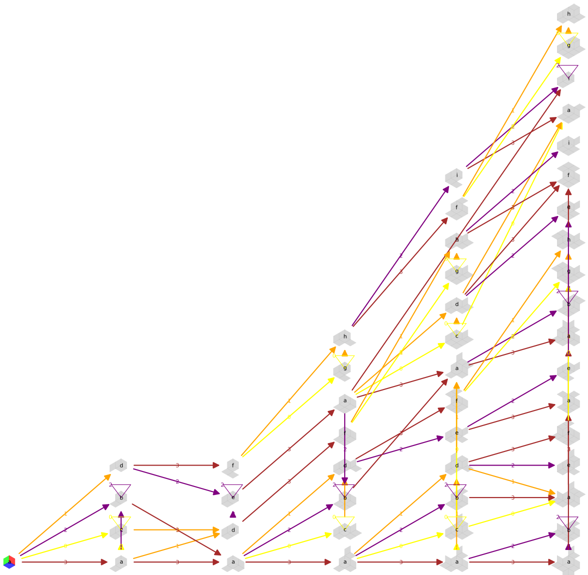

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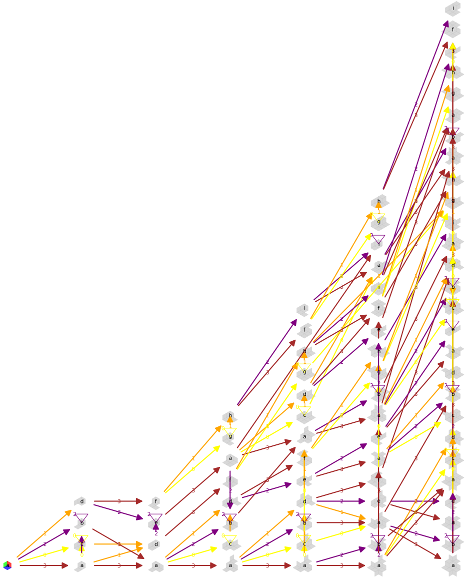

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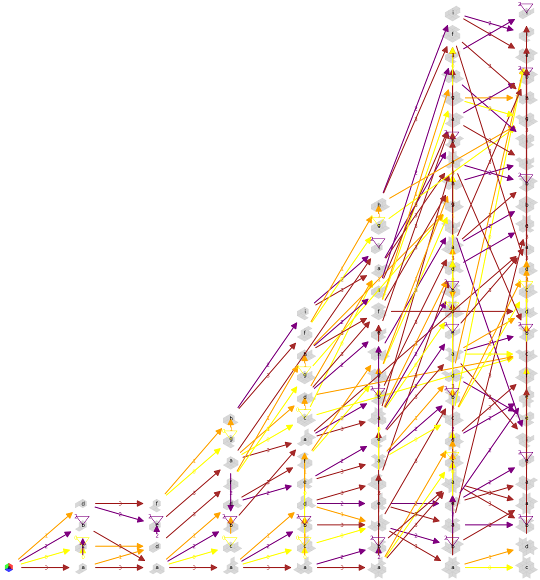
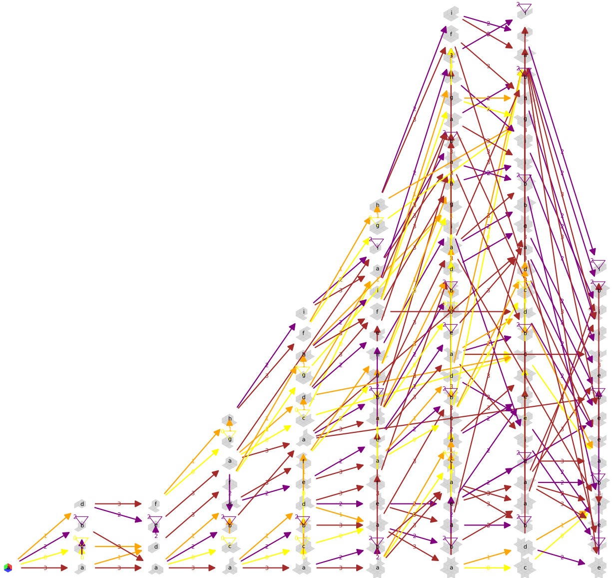
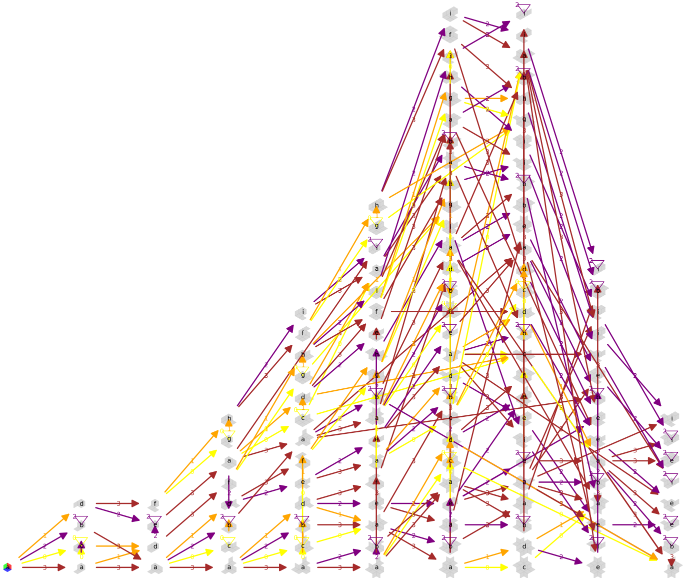


Nightmare (Jacobi-Perron)



Nightmare (Jacobi-Perron)



Nightmare (Jacobi-Perron)