Rauzy fractals with countable fundamental groups

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Joint work with **Benoît Loridant** and **Jun Luo**

Séminaire de Probabilités et Théorie Ergodique
LAMFA, Amiens
2014-01-07

Substitutions

$$\sigma : \begin{cases} 1 & \mapsto & 12 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 4 \\ 4 & \mapsto & 5 \\ 5 & \mapsto & 1 \end{cases} \qquad \mathbf{M}_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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Pisot substitution:

- Cubic Pisot eigenvalue $\beta \approx 1.325$, with $\beta', \beta'' \approx -0.662 \pm 0.562i$
- Two other eigenvalues $\approx 0.5 \pm 0.866i$

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Action of M_σ on $\mathbb{R}^5$:

- **Expanding line $\mathbb{E}$**

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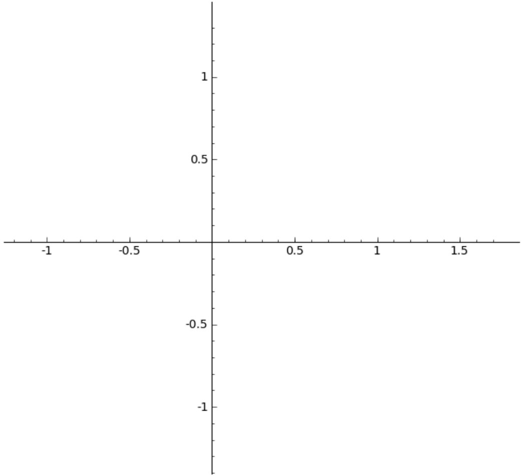
Action of M_σ on $\mathbb{R}^5$:

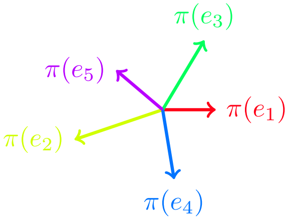
- **Expanding line** $\mathbb{E}$
- **Contracting plane** $\mathbb{P}$
- (Supplementary space $\mathbb{H}$)

Rauzy fractal of $1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 1$

$1234511212312341234512345112 3$



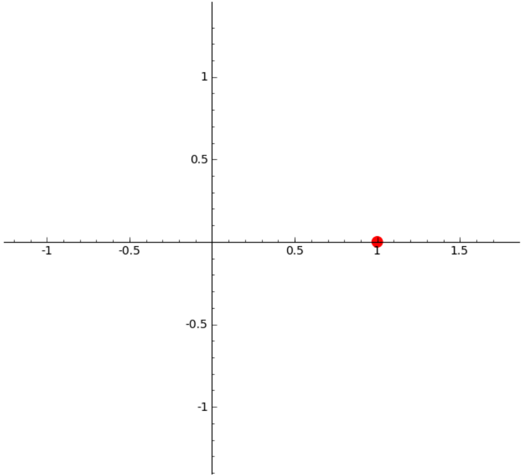
π = projection from $\mathbb{R}^5$ to $\mathbb{P}$ along $\mathbb{E} \oplus \mathbb{H}$



Rauzy fractal of $1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 1$

$123451121231234123451234511 23$

$\pi(e_1)$



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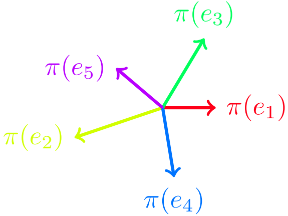
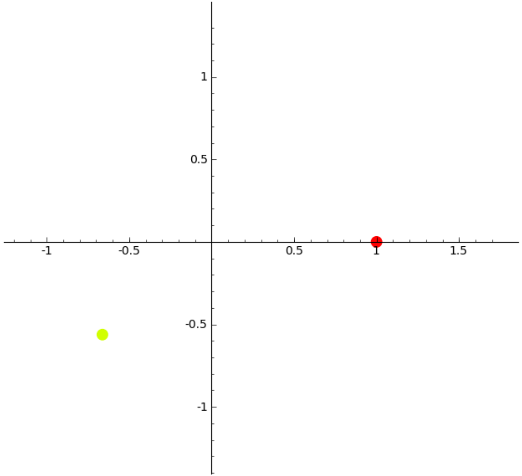


Rauzy fractal of $1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 1$

123451121231234123451234511123

$\pi(e_1) + \pi(e_2)$



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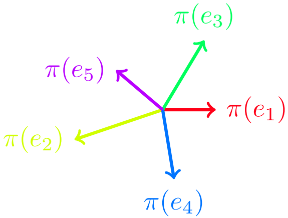

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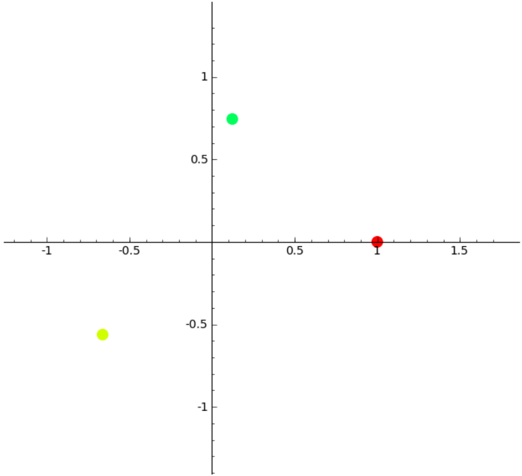
$\pi(e_1) + \pi(e_2) + \pi(e_3)$



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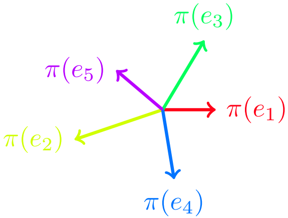

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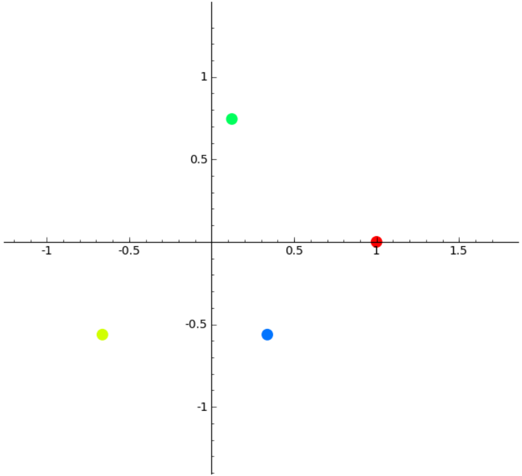
$\pi(e_1) + \pi(e_2) + \pi(e_3) + \pi(e_4)$



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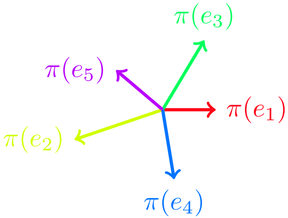

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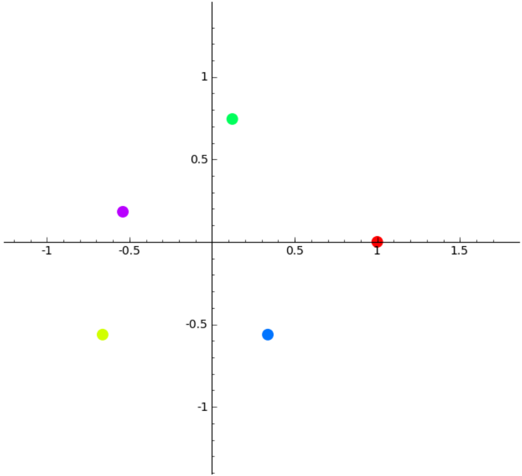

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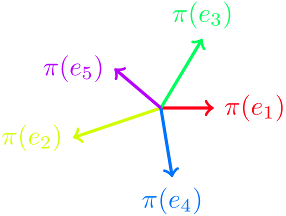

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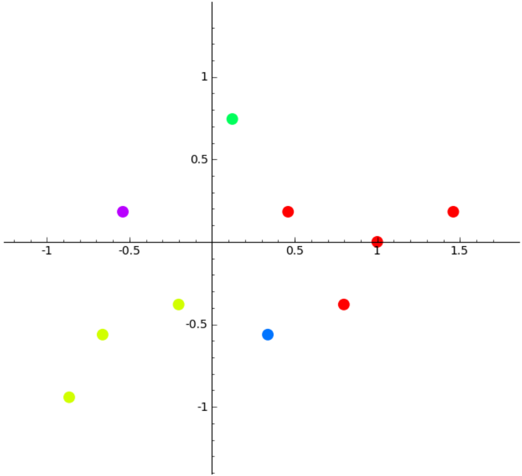

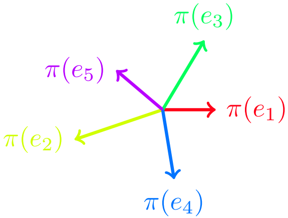

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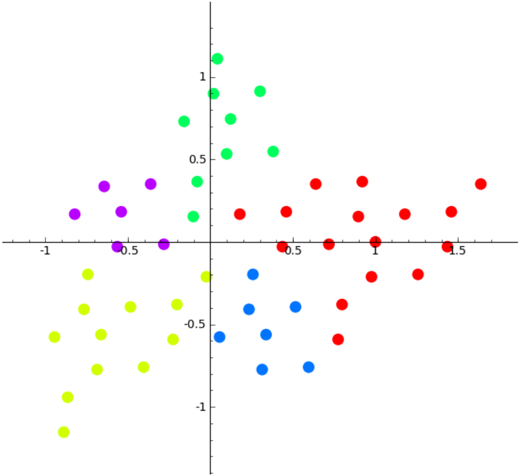

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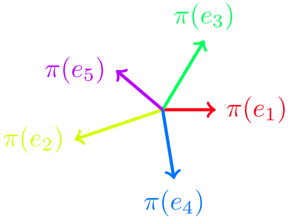

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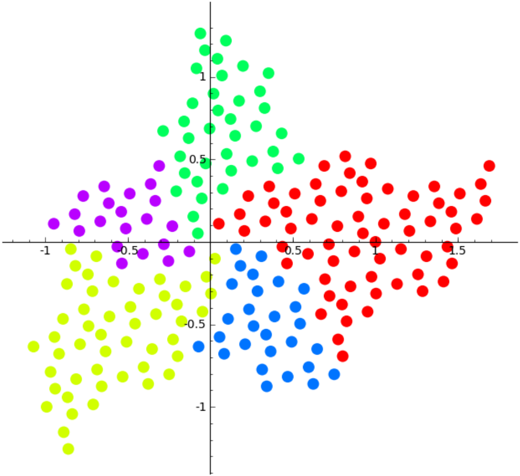

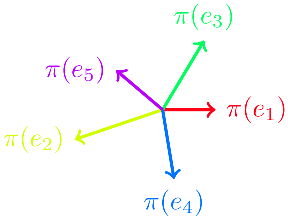

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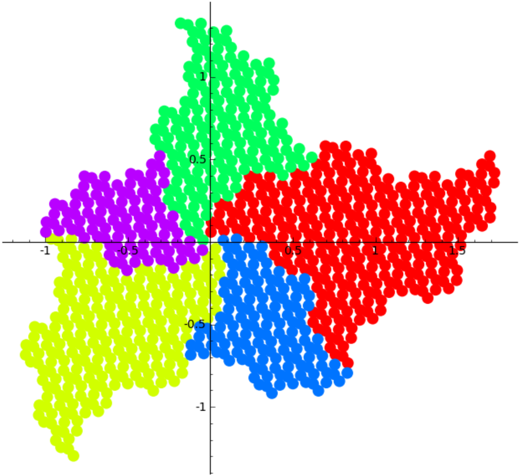

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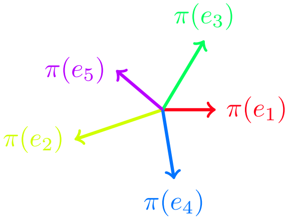

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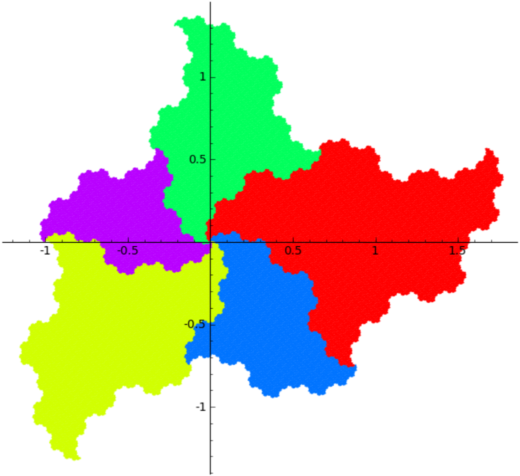
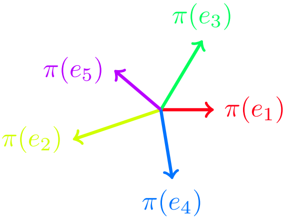
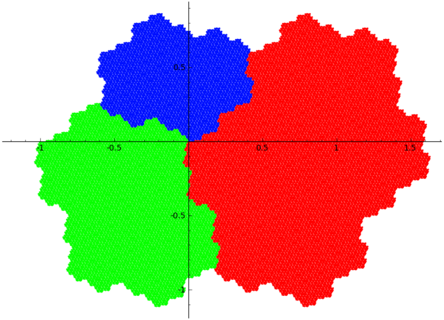




Other examples

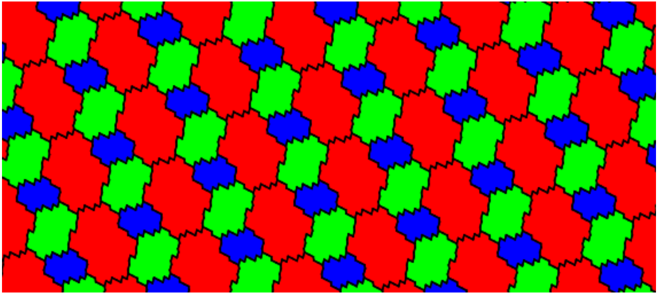
Fibonacci fractal $1 \mapsto 12, 2 \mapsto 1$:

Tribonacci fractal $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$:



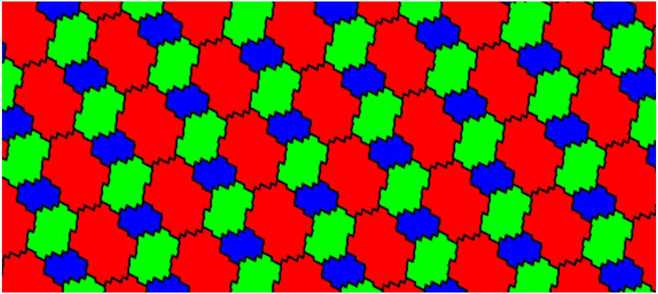
Main use: dynamics of Pisot substitutions

Periodic tiling



Main use: dynamics of Pisot substitutions

Periodic tiling $\longleftrightarrow$ partition of the torus $\mathbb{T}^2$



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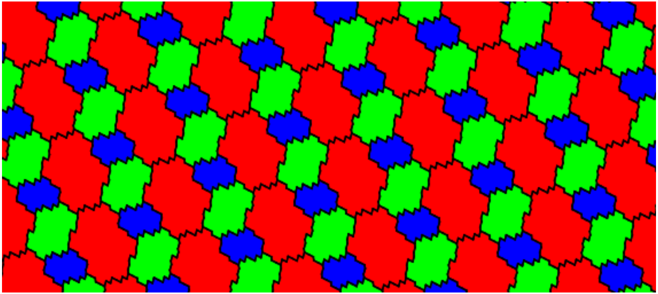

Allows to prove $(X_\sigma, \text{shift}) \cong (\mathbb{T}^2, x \mapsto x + (\frac{1}{\beta}, \frac{1}{\beta^2}))$

$$\ldots 1213121121 \ldots \in X_\sigma$$

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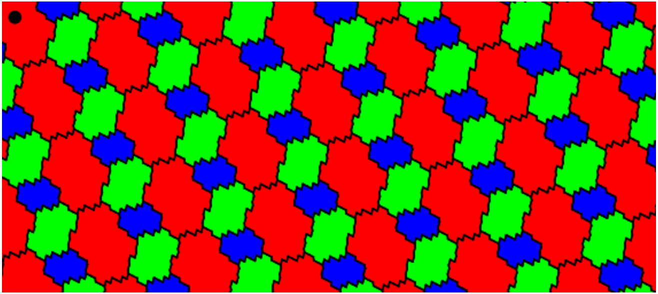

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$$\ldots \boxed{1}213121121 \ldots \in X_\sigma$$

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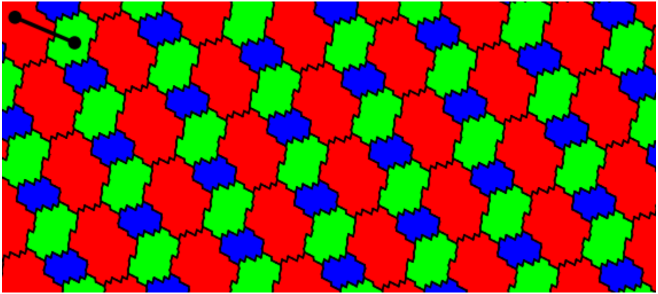

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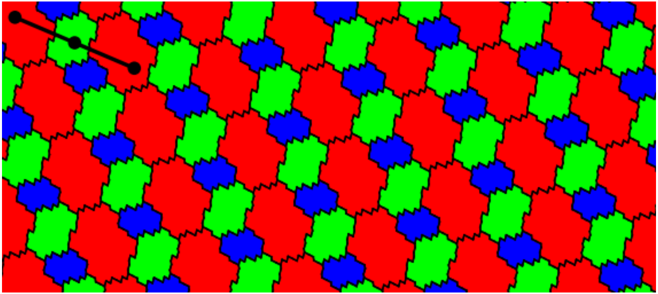

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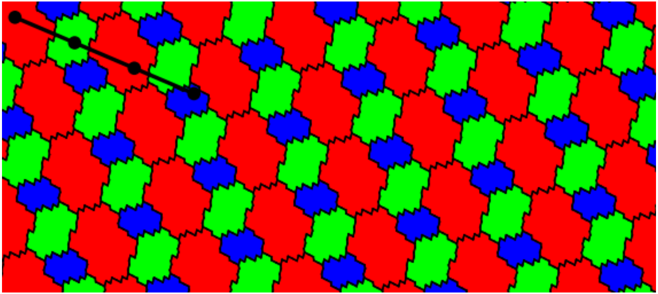

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$$\dots 121.3121121 \dots \in X_\sigma$$

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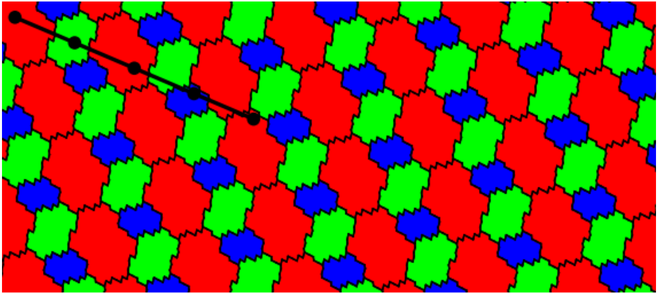
Periodic tiling $\longleftrightarrow$ partition of the torus $\mathbb{T}^2$



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$$\ldots 1213\boxed{1}121121 \ldots \in X_\sigma$$

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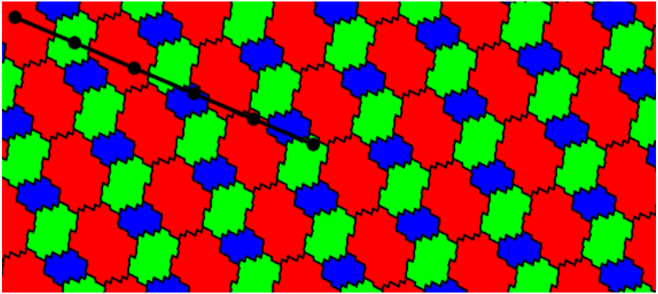

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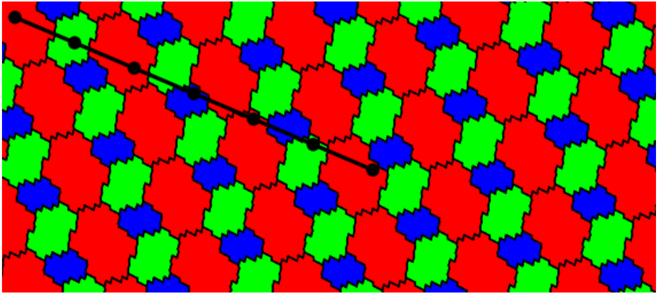

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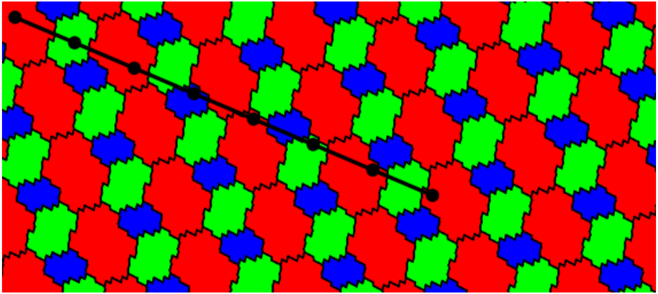

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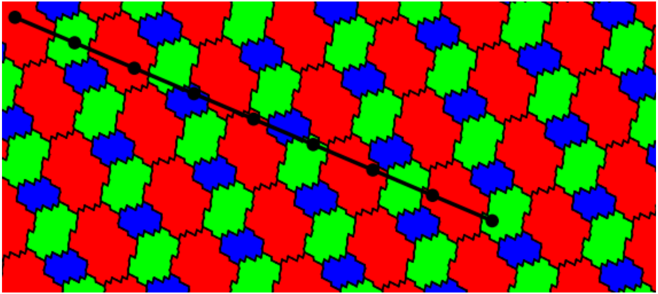

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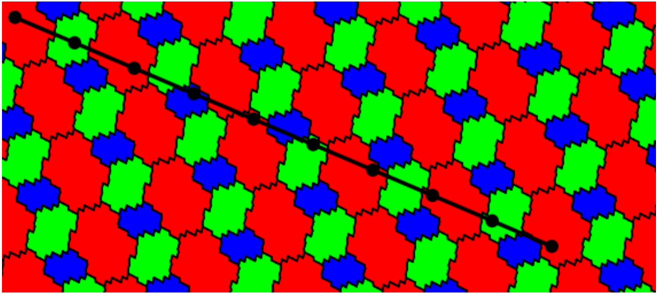

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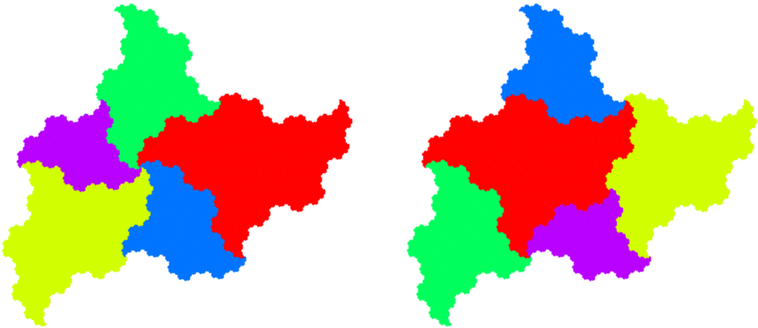
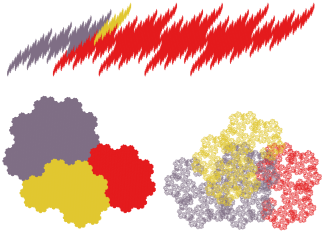
Basic facts about the tile

- **Topology:**
  - Graph-directed IFS, self-similar structure
  - Always: compact, locally connected, = closure(interior)
  - $\dim_H(\text{boundary}) \in]1, 2[$, computable
  - Tile intersections, connectedness, countability of the fund. group, zero is an inner point, ...: mostly **computable** [Siegel-Thuswaldner]
- Applications in several domains (dynamics, number theory, tiling spaces, ...)

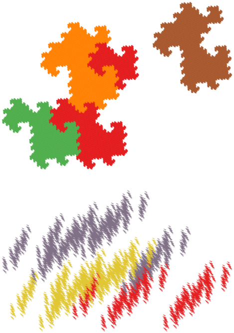


What kind of topological properties do we study?

- Cut points, connectedness, intersection of tiles, etc.



Connected



Not connected

What kind of topological properties do we study?

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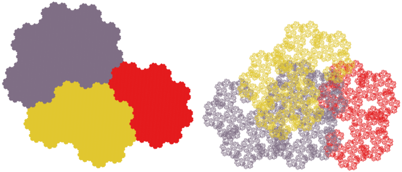
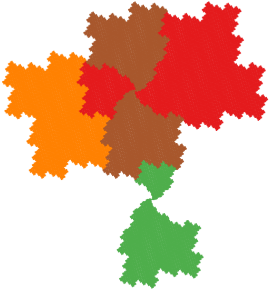


No cut points

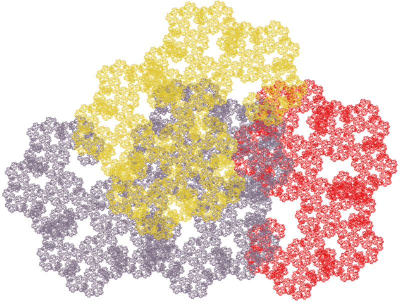


Cut points

What kind of topological properties do we study?

- Cut points, connectedness, intersection of tiles, etc.
- **Simple connectedness**

Disklike

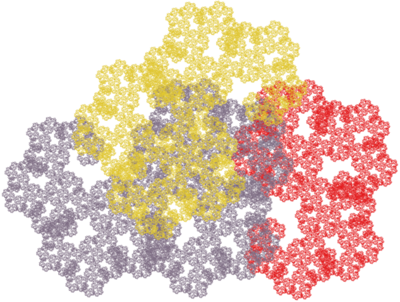


“Many holes”

What kind of topological properties do we study?

- Cut points, connectedness, intersection of tiles, etc.
- **Simple connectedness**: **fundamental group**

Disklike

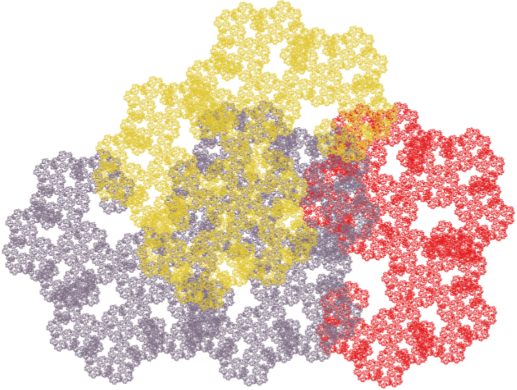


“Many holes”

Fundamental group, examples

Trivial

Fundamental group, examples



Uncountable, very hard to describe...

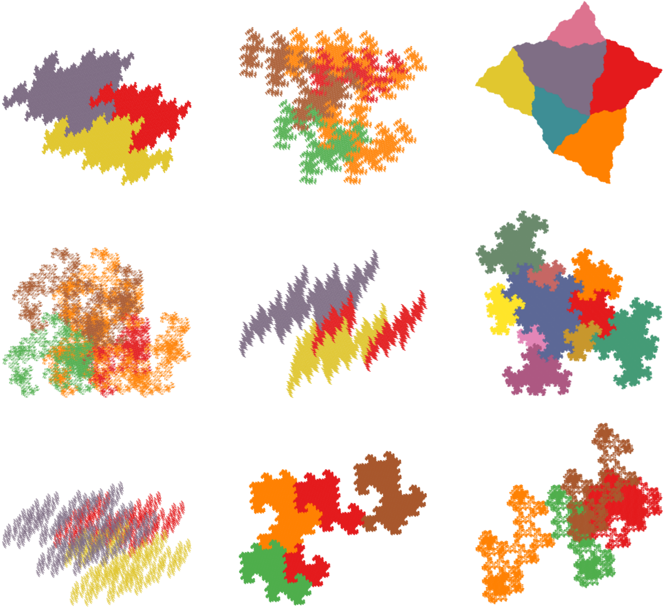
Fundamental group, examples

Doesn't look so bad, but. . .

Fundamental group, examples

Doesn't look so bad, but... uncountable

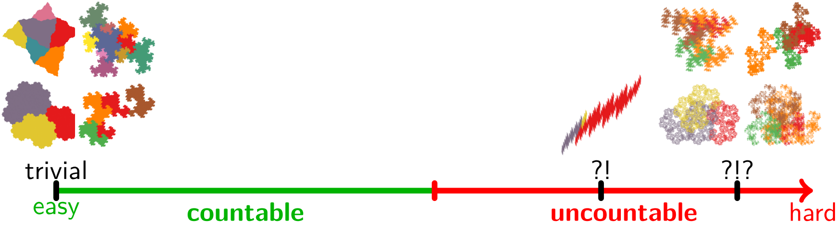
Fundamental group, examples



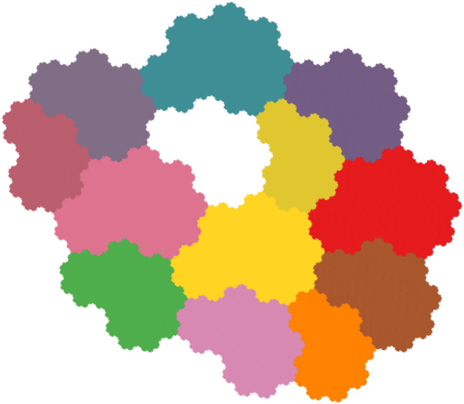
Fundamental group, examples

There seems to be a **dichotomy**:

trivial vs. uncountable



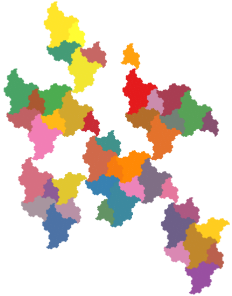
Example with nontrivial countable FG



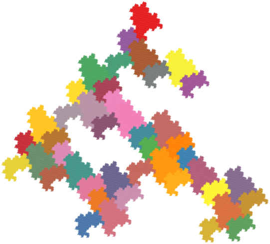
$$\pi_1(X) \cong F_1 \cong \mathbb{Z}$$

$$\begin{array}{rcl} 1 & \mapsto & 11, 1, 3, 8 \\ 3 & \mapsto & 5, 2, 4, 10, 12 \\ 5 & \mapsto & 5, 2, 4, 10, 12 \\ 7 & \mapsto & 6, 9, 7, 11, 1, 3, 8 \\ 9 & \mapsto & 11, 1, 3, 8 \\ 11 & \mapsto & 5, 2, 4, 10, 12 \end{array} \qquad \begin{array}{rcl} 2 & \mapsto & 11, 1, 3, 8 \\ 4 & \mapsto & 6, 9, 7 \\ 6 & \mapsto & 5, 2, 4, 10, 12 \\ 8 & \mapsto & 6, 9, 7, 11, 1, 3, 8 \\ 10 & \mapsto & 11, 1, 3, 8 \\ 12 & \mapsto & 5, 2, 4, 10, 12, 6, 9, 7, 11, 1, 3, 8. \end{array}$$

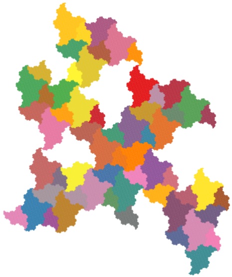
Other examples with nontrivial countable FG



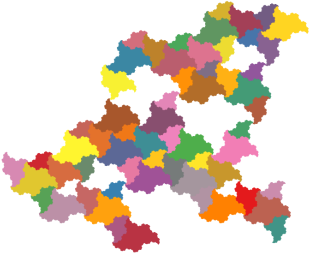
F_1



F_1



F_2



F_6

Countable FGs of Rauzy fractals, in general

Proposition

1. A nontrivial countable Rauzy fractal FG is always $\cong F_k$ ($k \in \mathbb{N}$).
2. k can be computed if all the tiles are discs (and intersect well).

Countable FGs of Rauzy fractals, in general

Proposition

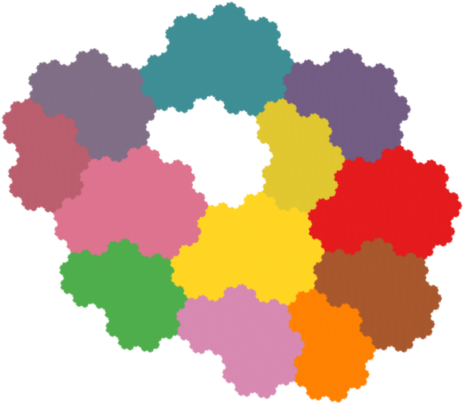
1. A nontrivial countable Rauzy fractal FG is always $\cong F_k$ ($k \in \mathbb{N}$).
2. k can be computed if all the tiles are discs (and intersect well).

Proof of 1: [Shelah 88]:
K compact, path-connected, locally path-connected metric space
$\Longrightarrow$ if $\pi_1(K)$ is not finitely generated then it is uncountable

(Similar proof also follows from [Conner-Lamoreaux 05])

How to prove that π_1 () $\cong F_1$?

- Prove that the 12 tiles are disklike
  and compute their neighboring graph [Siegel-Thuswaldner 2009]



How to prove that $\pi_1\Big($  $\Big) \cong F_1$?

- Prove that the 12 tiles are disklike
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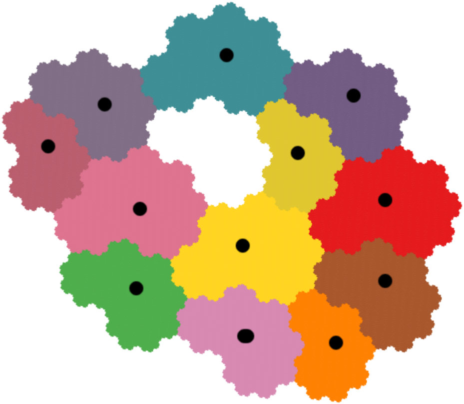
- Compute a topological complex with the same homotopy type



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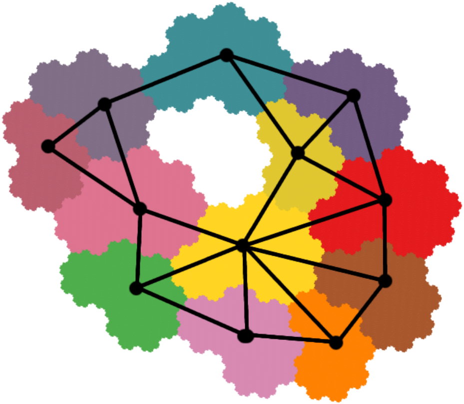

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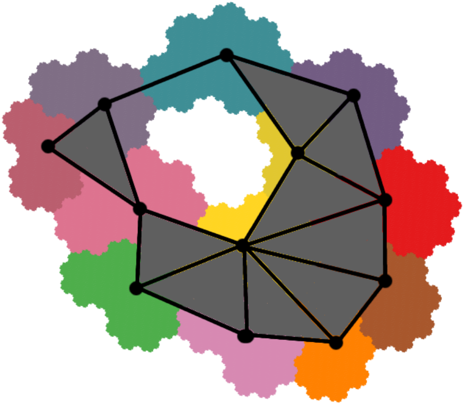

Question

- Previous examples F_1, F_2, F_6, found by experiment, trial and error

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- Can we get F_k for **every** $k \in \mathbb{N}$?

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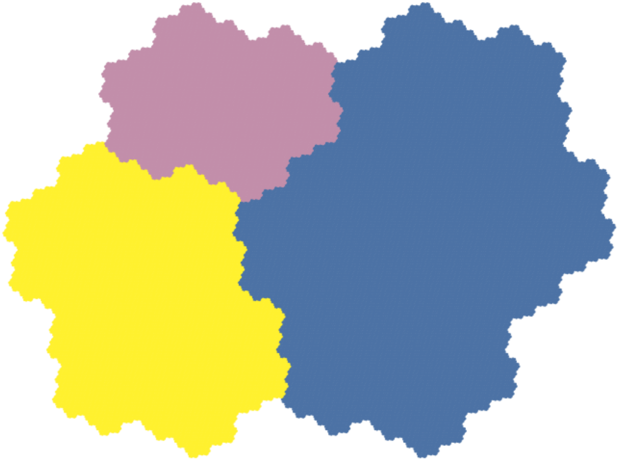
- Previous examples F_1, F_2, F_6, found by experiment, trial and error
- Can we get F_k for **every** $k \in \mathbb{N}$?
- **Difficulty:** how does modifying σ affect $\mathcal{T}_\sigma$?

Can we get all F_k? First approach: drill holes

- **Step 1:** subdivide the tiles



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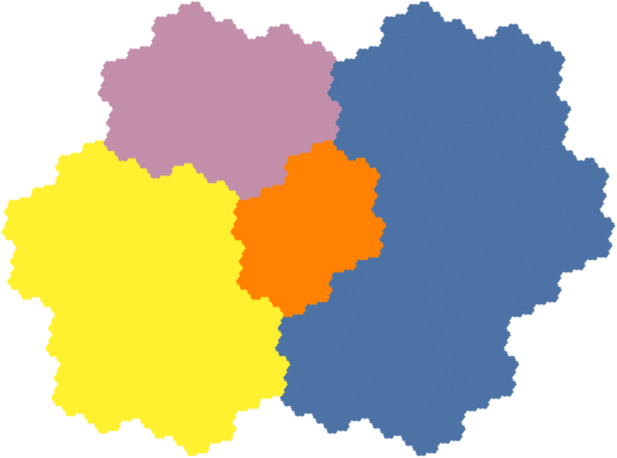

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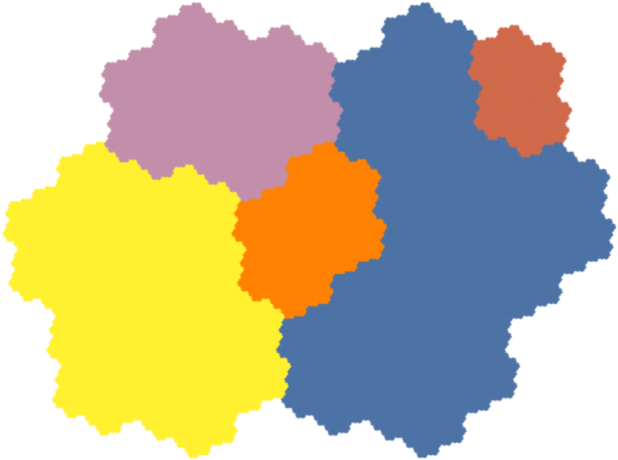

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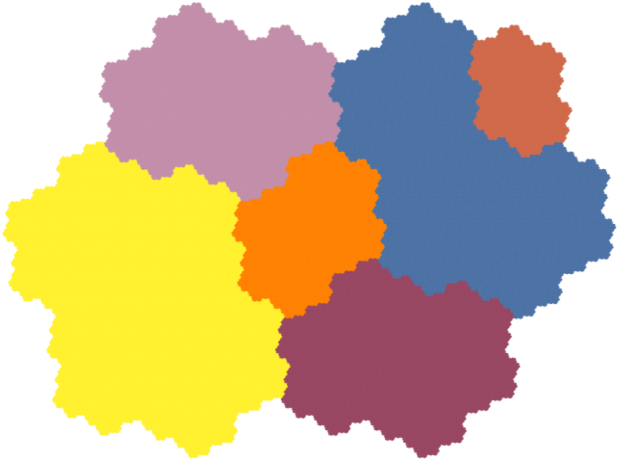

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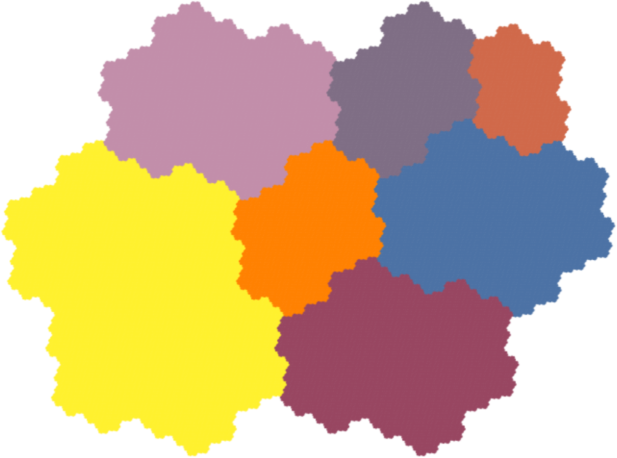

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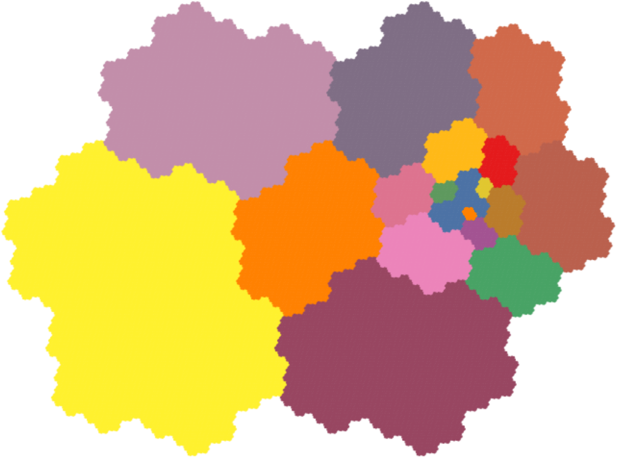
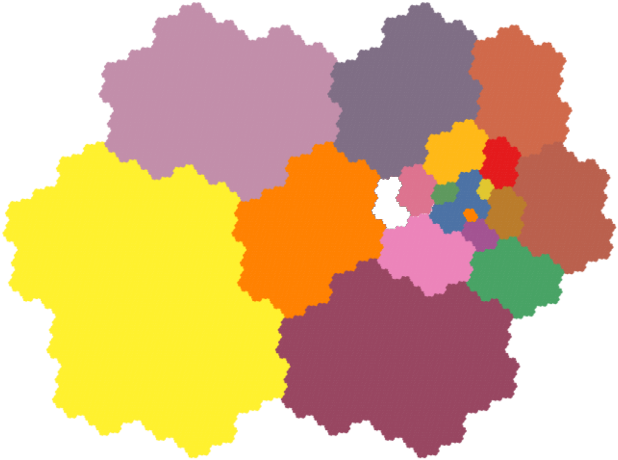


Can we get all F_k? First approach: drill holes

- **Step 1:** subdivide the tiles
- **Step 2:** shrink/move some tiles



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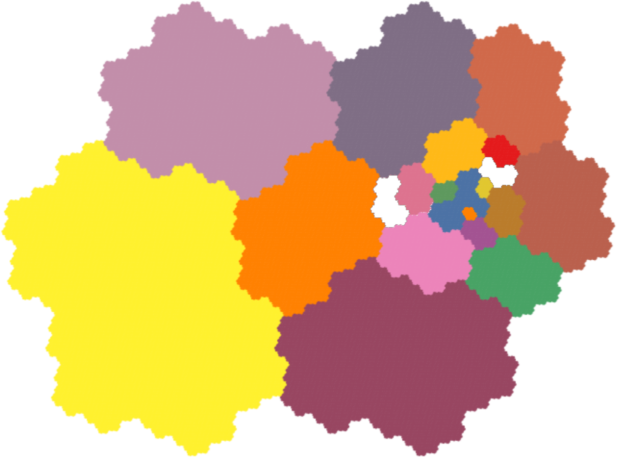

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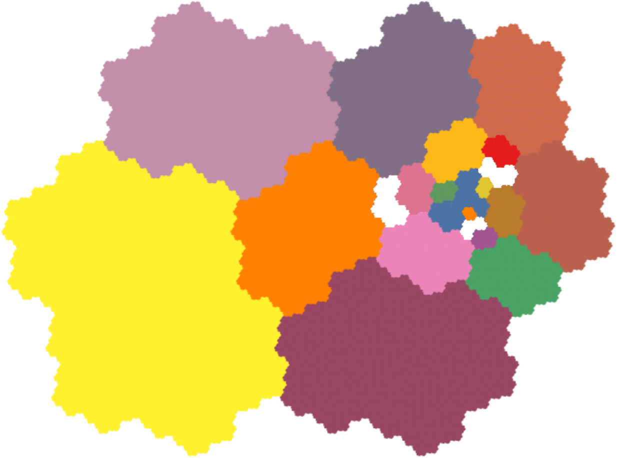
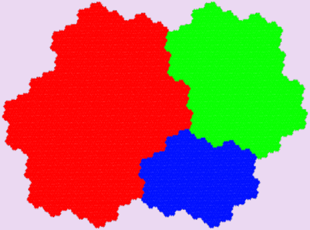


Free group automorphisms $\leftrightarrow$ Rauzy fractals

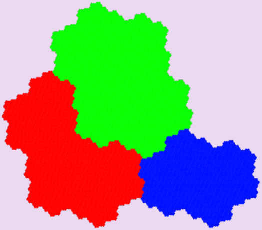
Example

$\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ $\qquad$ $\rho : 1 \mapsto 1, 2 \mapsto 12, 3 \mapsto 3$

$$\rho^{-1}\sigma\rho : \begin{cases} 1 \mapsto 1 \mapsto 12 \mapsto 1^{-1}12 = 2 \\ 2 \mapsto 12 \mapsto 1213 \mapsto 1^{-1}1213 = 213 \\ 3 \mapsto 3 \mapsto 1 \mapsto 1 = 1 \end{cases}$$



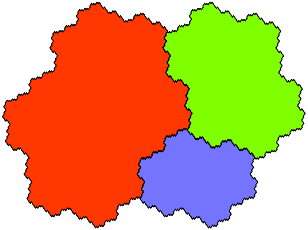
σ



$\rho^{-1}\sigma\rho$

We can refine this approach to get a general strategy.

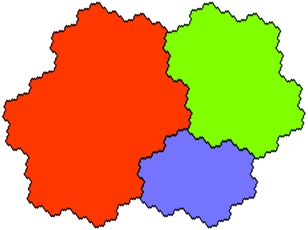
Step 1. Choose a good σ



$$\sigma : \begin{cases} 1 \mapsto 21 \\ 2 \mapsto 31 \\ 3 \mapsto 1 \end{cases}$$

- The fractal and its subtiles are discs

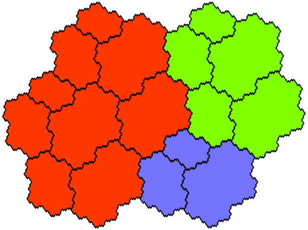
Step 2. Take powers of σ to subdivide tiles



$$\sigma^3 : \begin{cases} 1 \mapsto 1213121 \\ 2 \mapsto 213121 \\ 3 \mapsto 3121 \end{cases}$$

- subtiles of σ^3

Step 2. Take powers of σ to **subdivide** tiles



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- **Sub**subtiles of σ^3

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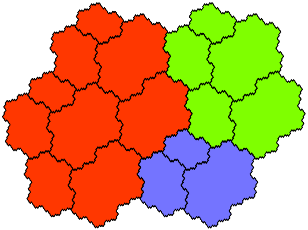

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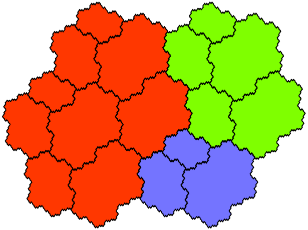
Definition

- $(j;k)$ is an **occurrence** of i if $\sigma(j)_k = i$
- occurrences $\Longleftrightarrow$ subsubtiles

Step 2. Take powers of σ to **subdivide** tiles



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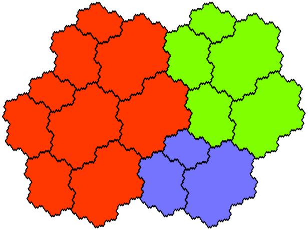
Definition

- $(j;k)$ is an **occurrence** of i if $\sigma(j)_k = i$
- occurrences $\Longleftrightarrow$ subsubtiles
- Example:

$$\mathrm{occ}(\sigma^3, 1) = \{(1,1),(1,3),(1,5),(1,7),(2,2),(2,4),(2,6),(3,2),(3,4)\}$$
$$\mathrm{occ}(\sigma^3, 2) = \{(1,2),(1,6),(2,1),(2,5),(3,3)\}$$
$$\mathrm{occ}(\sigma^3, 3) = \{(1,4),(2,3),(3,1)\}$$

Step 2. Take powers of σ to **subdivide** tiles



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Graph-IFS decomposition [Sirvent-Wang 02]

- Subtile decomposition $\mathcal{T}_\sigma = \mathcal{T}_\sigma(1) \cup \mathcal{T}_\sigma(2) \cup \mathcal{T}_\sigma(3)$
- Subsubtile decompostition $\mathcal{T}_\sigma(i) = \bigcup_{o \in \mathrm{occ}(\sigma, i)} \mathcal{T}_\sigma(i, j; k)$

  (with **subsubtile** $= \mathcal{T}_\sigma(i, j; k) = \mathbf{h}_\sigma \mathcal{T}_\sigma(j) + \pi_\sigma(\sigma(j)_{[1,\ldots,k-1]})$)

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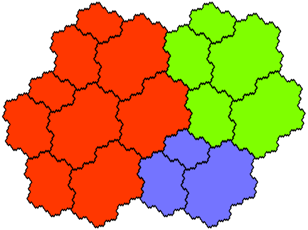

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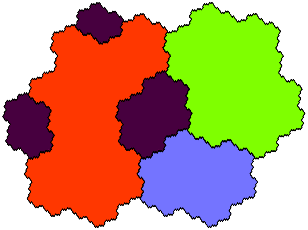
➥ **We will now remove some subsubtiles**

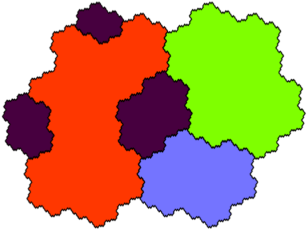
Step 3. Split a symbol to **isolate** tiles



$$\tau : \begin{cases} 1 \mapsto 1213121 \\ 2 \mapsto 213121 \\ 3 \mapsto 3121 \\ \end{cases}$$

- State splitting

Step 3. Split a symbol to **isolate** tiles



$$\tau : \begin{cases} 1 \mapsto 4213121 \\ 2 \mapsto 2131 24 \\ 3 \mapsto 3421 \\ \end{cases}$$

- State splitting

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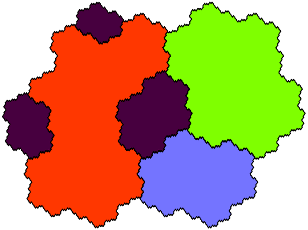
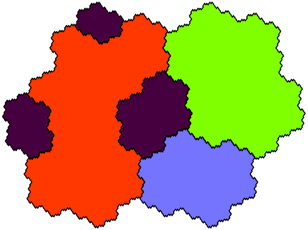


$$\tau : \begin{cases} 1 \mapsto 4213121 \\ 2 \mapsto 213124 \\ 3 \mapsto 3421 \\ 4 \mapsto 4213121 \end{cases}$$

- State splitting

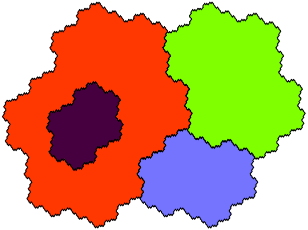
Step 3. Split a symbol to isolate tiles



$$\tau : \begin{cases} 1 \mapsto 4213121 \\ 2 \mapsto 213124 \\ 3 \mapsto 3421 \\ 4 \mapsto 4213121 \end{cases}$$

- State splitting
- Subtiles $\mathcal{T}_\tau(1)$, $\mathcal{T}_\tau(2)$, $\mathcal{T}_\tau(3)$, $\mathcal{T}_\tau(4)$

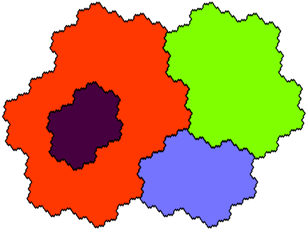
Step 3. Split a symbol to **isolate** tiles



Another state splitting

$$\tau : \begin{cases} 1 \mapsto 1213121 \\ 2 \mapsto 213121 \\ 3 \mapsto 3121 \\ \end{cases}$$

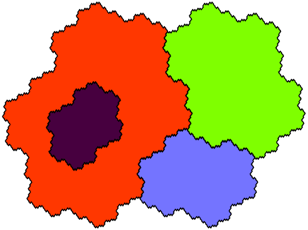
Step 3. Split a symbol to **isolate** tiles



Another state splitting

$$\tau : \begin{cases} 1 \mapsto 1213124 \\ 2 \mapsto 213121 \\ 3 \mapsto 3121 \\ \end{cases}$$

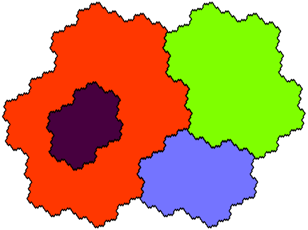
Step 3. Split a symbol to **isolate** tiles



Another state splitting

$$\tau : \begin{cases} 1 \mapsto 1213124 \\ 2 \mapsto 213121 \\ 3 \mapsto 3121 \\ 4 \mapsto 1213124 \end{cases}$$

Step 3. Split a symbol to **isolate** tiles



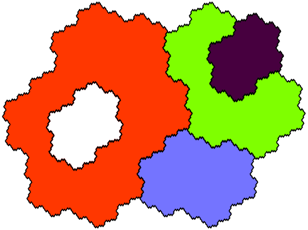
$$\tau : \begin{cases} 1 \mapsto 1213124 \\ 2 \mapsto 213121 \\ 3 \mapsto 3121 \\ 4 \mapsto 1213124 \end{cases}$$

Proposition: effect of splitting on $\mathcal{T}_\sigma$

Let $\tau =$ splitting of occurrences $I \subseteq \mathrm{occ}(\sigma, 1)$ of 1 to 4

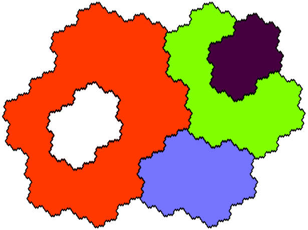
- $\mathcal{T}_\tau(2) = \mathcal{T}_\sigma(2)$
- $\mathcal{T}_\tau(3) = \mathcal{T}_\sigma(3)$
- $\mathcal{T}_\tau(4) = \bigcup_{(j;k) \in I} \mathcal{T}_\sigma(1, j; k)$.
- $\mathcal{T}_\tau(1) = \bigcup_{(j;k) \in \mathrm{occ}(\sigma,1) \setminus I} \mathcal{T}_\sigma(1, j; k)$,

Step 4. Conjugate by a free group aut. to **move** tiles



$$\rho_{24}^{-1}\tau\rho_{24} : \begin{cases} 1 \mapsto 1 \mapsto 1213124 \mapsto 1213122 2^{-1}4 \\ 2 \mapsto 2 \mapsto 213121 \mapsto 213121 \\ 3 \mapsto 3 \mapsto 3121 \mapsto 3121 \\ 4 \mapsto 24 \mapsto 2131211213124 \mapsto 21312112131 2 2^{-1}4 \end{cases}$$

Step 4. Conjugate by a free group aut. to **move** tiles



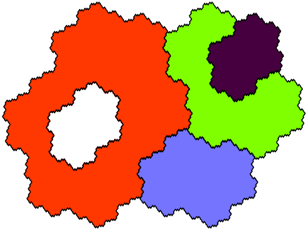
$$\rho_{24}^{-1}\tau\rho_{24} : \begin{cases} 1 \mapsto 12131\,\cancel{2}\,4 \\ 2 \mapsto 213121 \\ 3 \mapsto 3121 \\ 4 \mapsto 21312112131\,\cancel{2}\,4 \end{cases}$$

Proposition: effect of free group aut. conjugation on $\mathcal{T}_\tau$

Let $\theta = \rho_{24}^{-1}\tau\rho_{24}$

- $\mathcal{T}_\theta(1) = \mathcal{T}_\tau(1)$
- $\mathcal{T}_\theta(3) = \mathcal{T}_\tau(3)$
- $\mathcal{T}_\theta(2) \cup \mathcal{T}_\theta(4) = \mathcal{T}_\tau(2)$

Step 4. Conjugate by a free group aut. to **move** tiles



$$\rho_{24}^{-1}\tau\rho_{24} : \begin{cases} 1 \mapsto 12131\,\cancel{2}\,4 \\ 2 \mapsto 213121 \\ 3 \mapsto 3121 \\ 4 \mapsto \color{red}{213121}12131\,\cancel{2}\,4 \end{cases}$$

Proposition: effect of free group aut. conjugation on $\mathcal{T}_\tau$

Let $\theta = \rho_{24}^{-1}\tau\rho_{24}$

- $\mathcal{T}_\theta(1) = \mathcal{T}_\tau(1)$
- $\mathcal{T}_\theta(3) = \mathcal{T}_\tau(3)$
- $\mathcal{T}_\theta(2) \cup \mathcal{T}_\theta(4) = \mathcal{T}_\tau(2)$

➥ **This is now enough**

Countable FGs of Rauzy fractals, in general

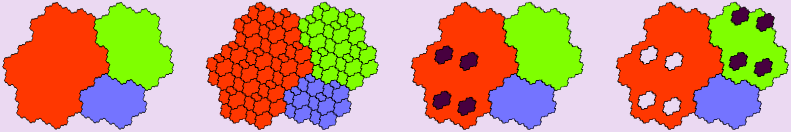
Theorem 2 [J-Loridant-Luo]

For **every** $k \in \mathbb{N}$, there exists σ such that

- $\pi_1(\mathcal{T}_\sigma(1)) \cong F_k$
- $\pi_1(\mathcal{T}_\sigma) \cong F_k$

Moreover, σ can be taken with 4 letters only.

$k = 4$



σ $\quad$ σ^6 $\quad$ splitting τ $\quad$ $\rho_{42}^{-1}\tau\rho_{42}$

Countable FGs of Rauzy fractals, in general

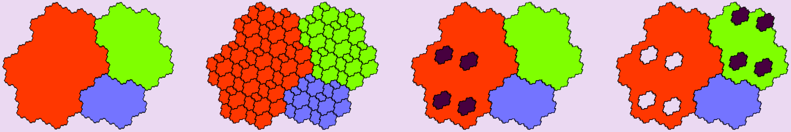
Theorem 2 [J-Loridant-Luo]

For **every** $k \in \mathbb{N}$, there exists σ such that

- $\pi_1(\mathcal{T}_\sigma(1)) \cong F_k$
- $\pi_1(\mathcal{T}_\sigma) \cong F_k$

Moreover, σ can be taken with 4 letters only.

$k = 4$

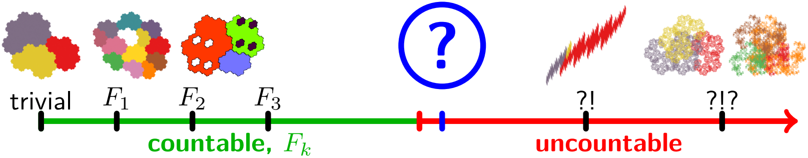


σ | σ^6 | splitting τ | $\rho_{42}^{-1}\tau\rho_{42}$

➥ **Remark: FG is not preserved by free group auts.**

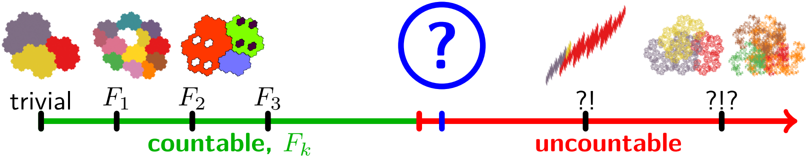
(related with questions of Gähler, Arnoux-Berthé-Hilion-Siegel)

Uncountable, but manageable FG?



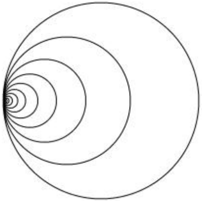
A Rauzy fractal with **uncountable** FG that we can **describe in detail**?

Uncountable, but manageable FG?

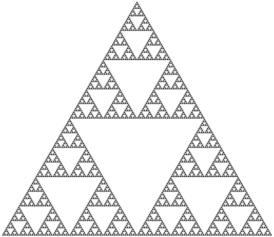


A Rauzy fractal with **uncountable** FG that we can **describe in detail**?

Like:

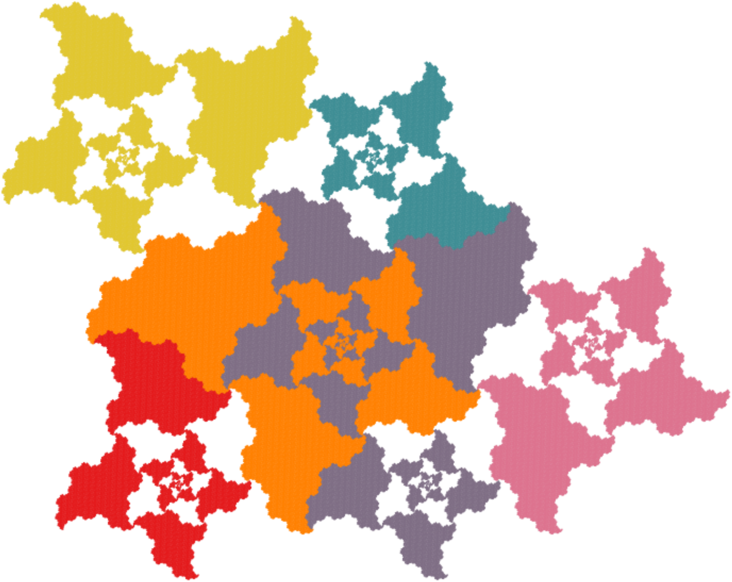


Hawaiian Earring
(Cannon-Conner 2000)

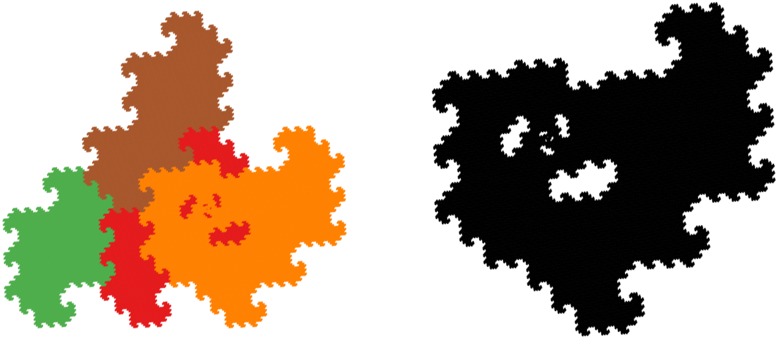


Sierpiński gasket
(Akiyama-Dorfer
-Thuswaldner-Winkler 2009)

Uncountable, but manageable FG? — Candidate 1



Uncountable, but manageable FG? — Candidate 2



Summary

- Splittings + free group auts

  $\longleftrightarrow$

  Operations on rauzy fractals
- Tool for Rauzy fractal topology

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Questions

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- Use these tools in dynamics

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Merci de votre attention