Substitutions and connectedness of Rauzy fractals

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Joint work with Valérie Berthé and Anne Siegel

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Rauzy fractals

To a substitution $\sigma : \{1,2,3\}^* \to \{1,2,3\}^*$, we can associate a Rauzy fractal:

- Compact subset of $\mathbb{R}^2$ with fractal boundary
- Uses: geometric realizations of substitutions (originally);
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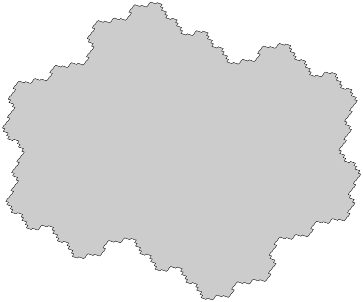
The Tribonacci substitution
$\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$



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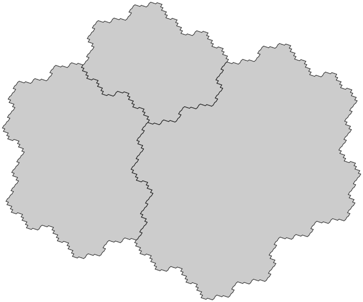

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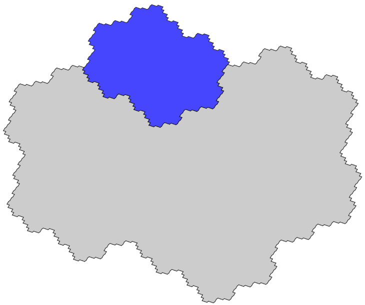

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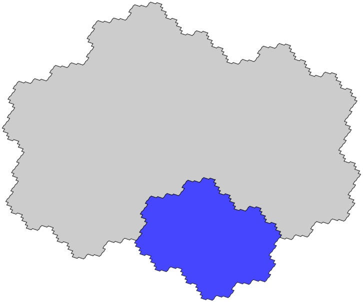

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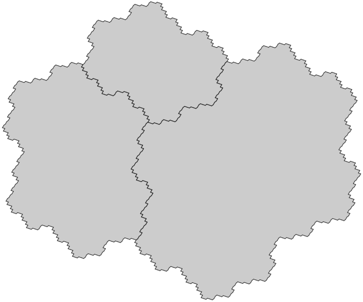

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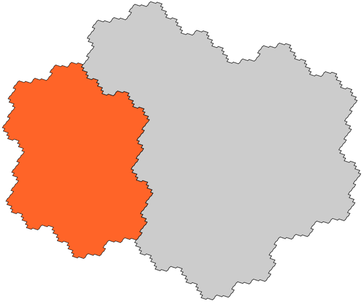

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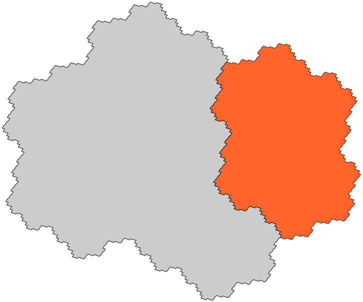

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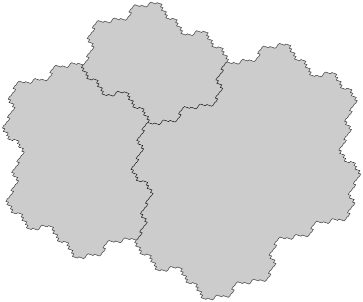

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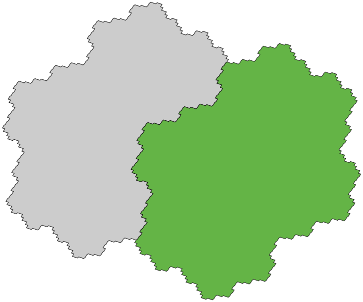

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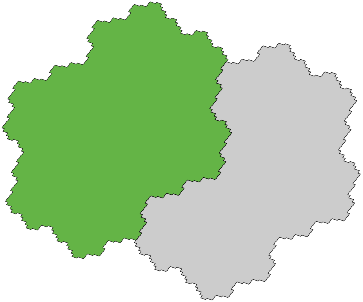

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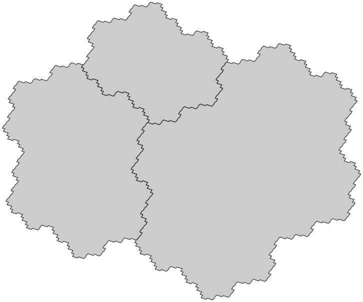

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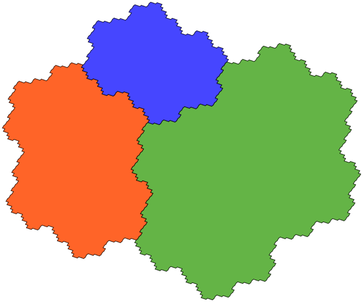

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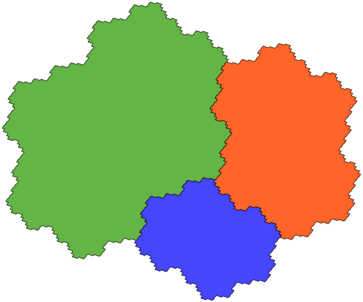
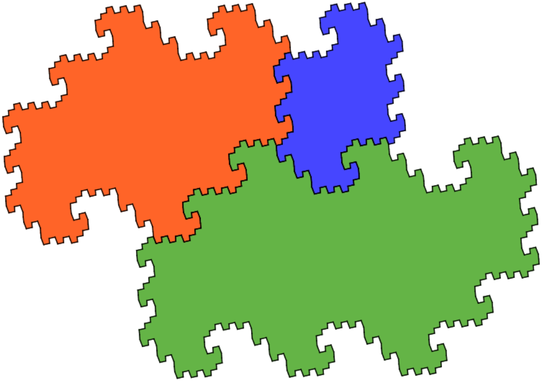


Dynamics of $\sigma : 1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$



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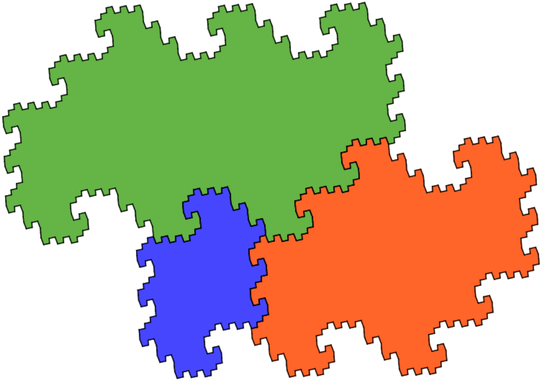

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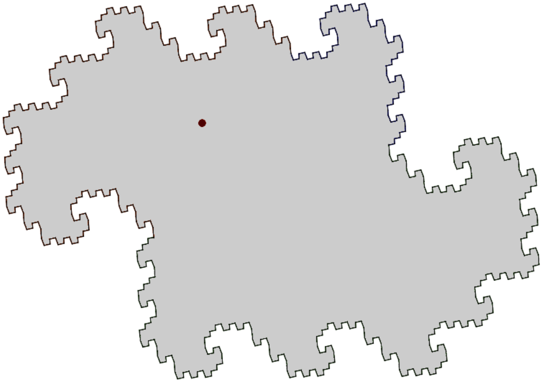

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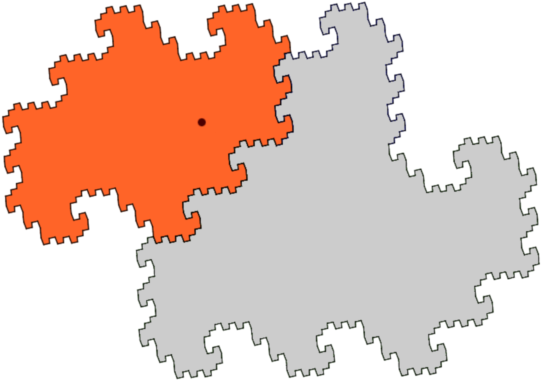

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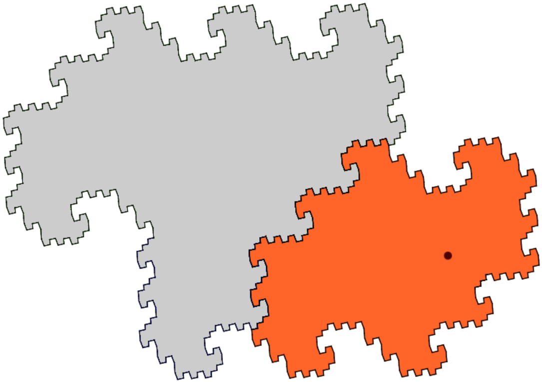

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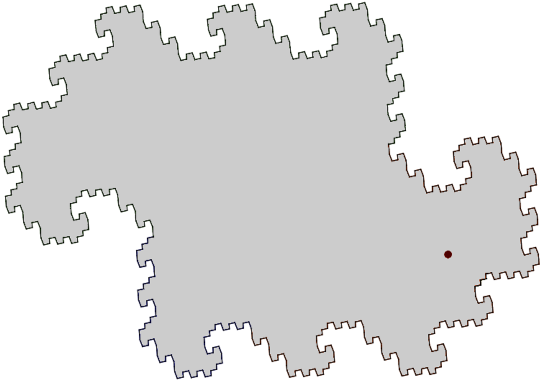
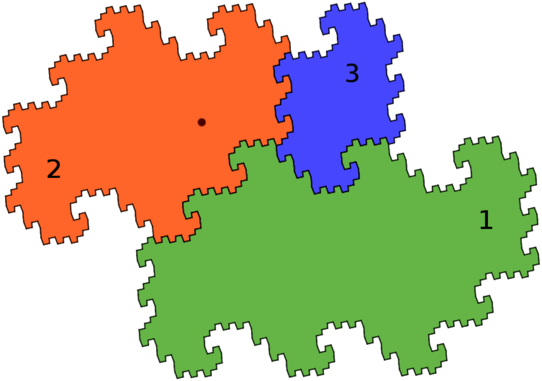


Dynamics of $\sigma : 1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$

Trajectory: 2



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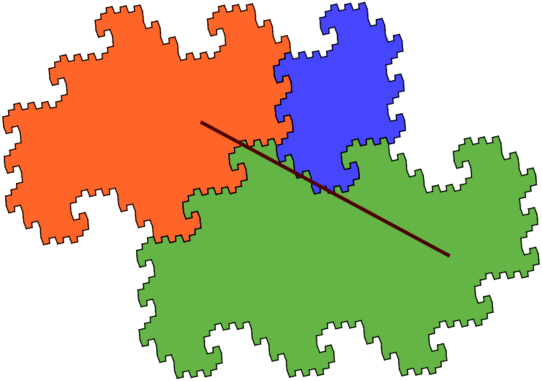
Trajectory: 21



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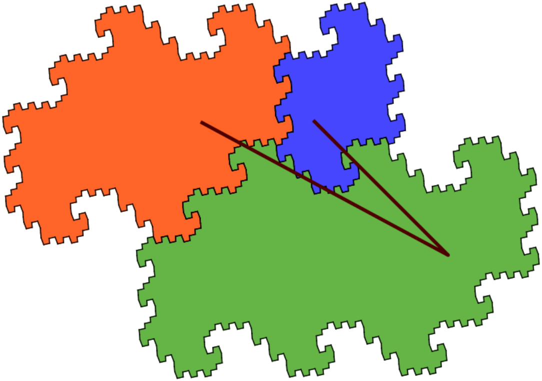
Trajectory: 213



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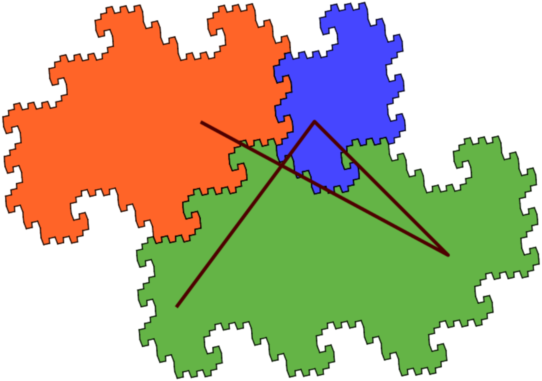

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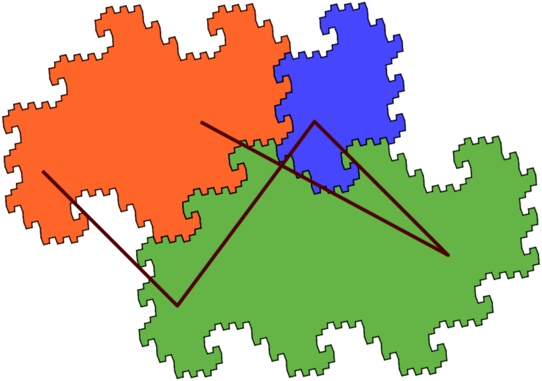
Trajectory: 21312



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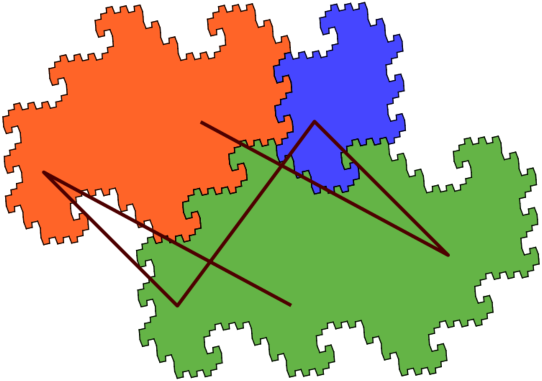

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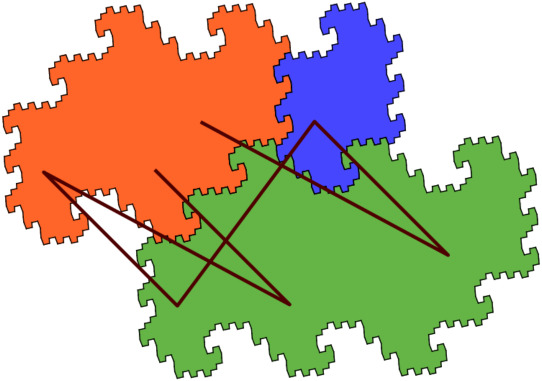
Trajectory: 2131212



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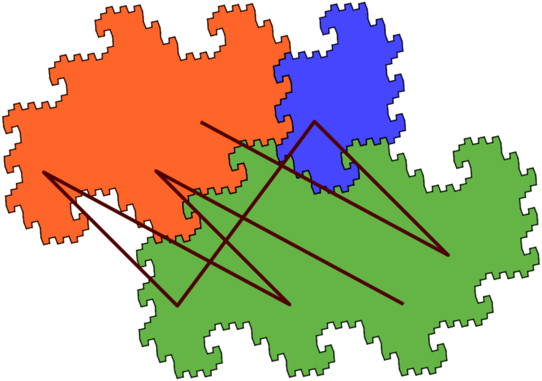
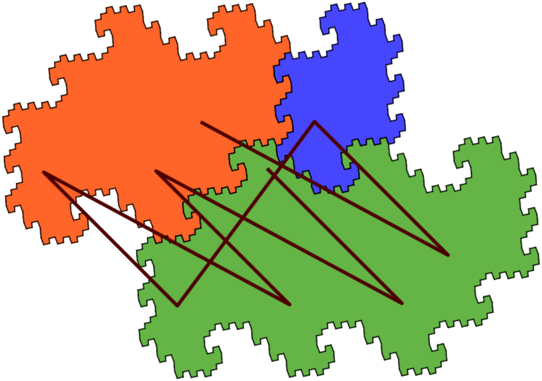
Trajectory: 21312121



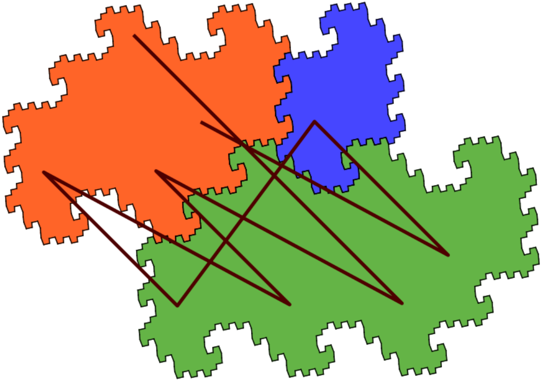
Dynamics of $\sigma : 1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$

Trajectory: 213121211



Dynamics of $\sigma : 1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$

Trajectory: 2131212112



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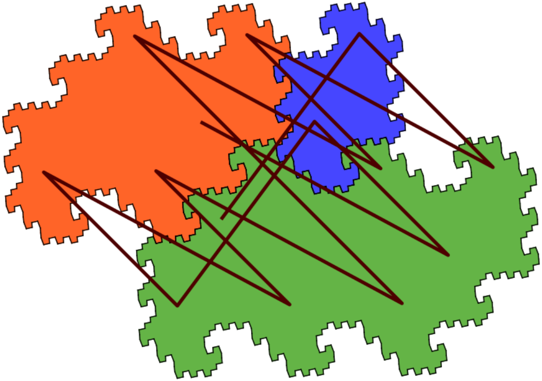
Trajectory: 21312121121212131



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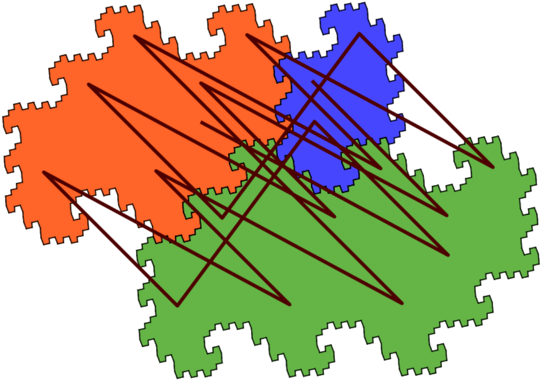
Trajectory: 2131212112121312121213



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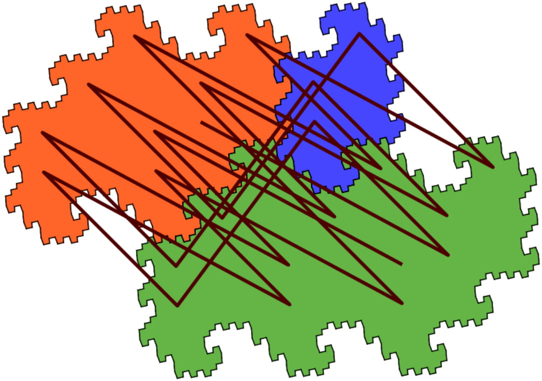
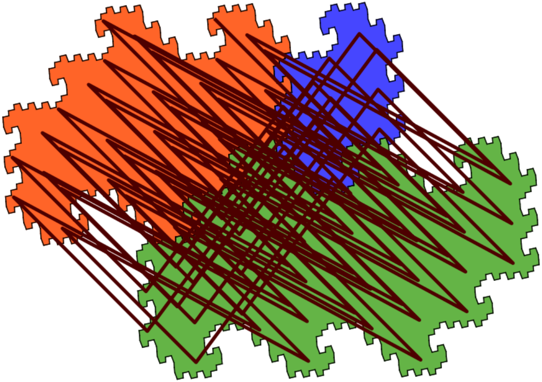


Dynamics of $\sigma : 1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$

Trajectory: $\cdots 213121211212131212131212 1 \cdots \in X_\sigma \subseteq \{1,2,3\}^{\mathbb{Z}}$



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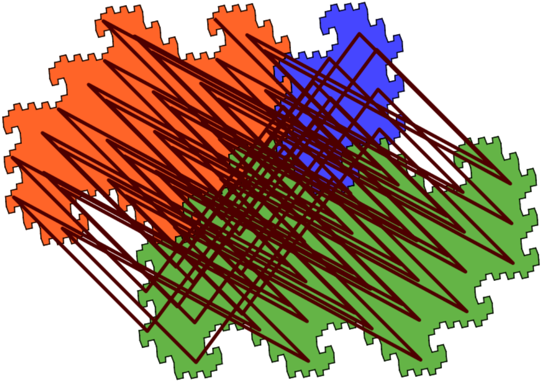
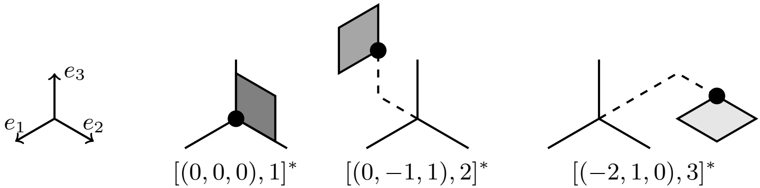


$\left(X_\sigma \right.$, shift$\left. \right) \cong \Big($  , domain exchange $\Big)$

Unit faces

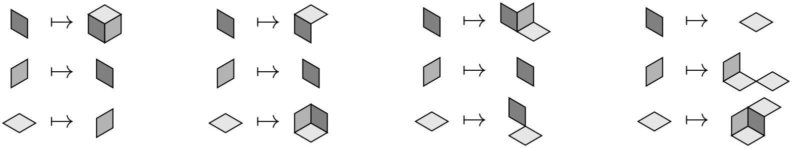
A **unit face** $[\mathbf{x}, i]^*$ is specified by:

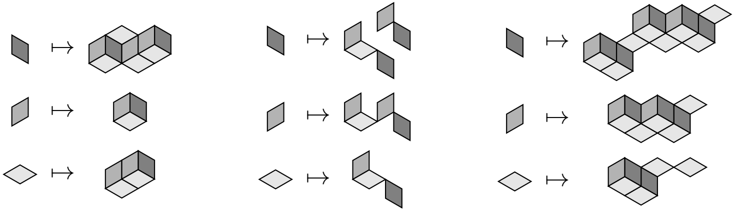
- its **position** $\mathbf{x} \in \mathbb{Z}^3$
- its **type** $i \in \{1, 2, 3\}$



Notation: $\mathbf{x} + D$: set of faces D translated by $\mathbf{x} \in \mathbb{Z}^3$

Some substitutions of unit faces





Definition: dual substitutions [Arnoux-Ito '01]

For unimodular $\sigma : \{1,2,3\}^* \to \{1,2,3\}^*$, let

$$\mathbf{E}_1^*(\sigma)([\mathbf{x}, i]^*) \ = \ \mathbf{M}_\sigma^{-1}\mathbf{x} \ + \bigcup_{k=1,2,3} \ \bigcup_{s | \sigma(k) = pis} [\ell(s), k]^*,$$

where:

- $\mathbf{M}_\sigma$ is the incidence matrix of σ;
- $\ell : \{1,2,3\}^* \to \mathbb{Z}_+^3$, $\qquad w \mapsto (|w|_1, |w|_2, |w|_3)$.

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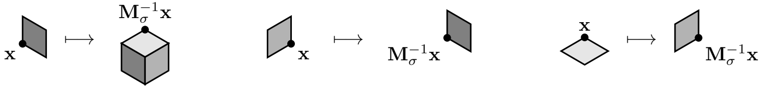
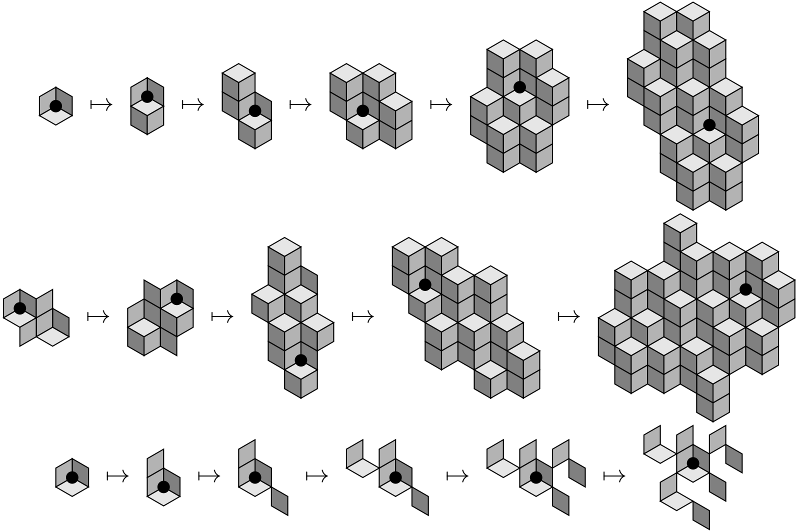
Example for $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$

$$\begin{aligned}
\mathbf{E}_1^*(\sigma)([\mathbf{x},1]^*) &= \mathbf{M}_\sigma^{-1}\mathbf{x} + [(1,0,-1),1]^* \cup [(0,1,-1),2]^* \cup [(0,0,0),3]^* \\
\mathbf{E}_1^*(\sigma)([\mathbf{x},2]^*) &= \mathbf{M}_\sigma^{-1}\mathbf{x} + [(0,0,0),1]^* \\
\mathbf{E}_1^*(\sigma)([\mathbf{x},3]^*) &= \mathbf{M}_\sigma^{-1}\mathbf{x} + [(0,0,0),2]^*
\end{aligned}$$



Examples



Properties

We remain within a discrete plane [Arnoux-Ito '01, Fernique '07]

$\forall n \geqslant 0, \quad \mathbf{E}_1^*(\sigma)^n($  $) \subseteq$ a discrete plane

(More generally, the image of a discrete plane by $\mathbf{E}_1^*(\sigma)$ is also a discrete plane.)

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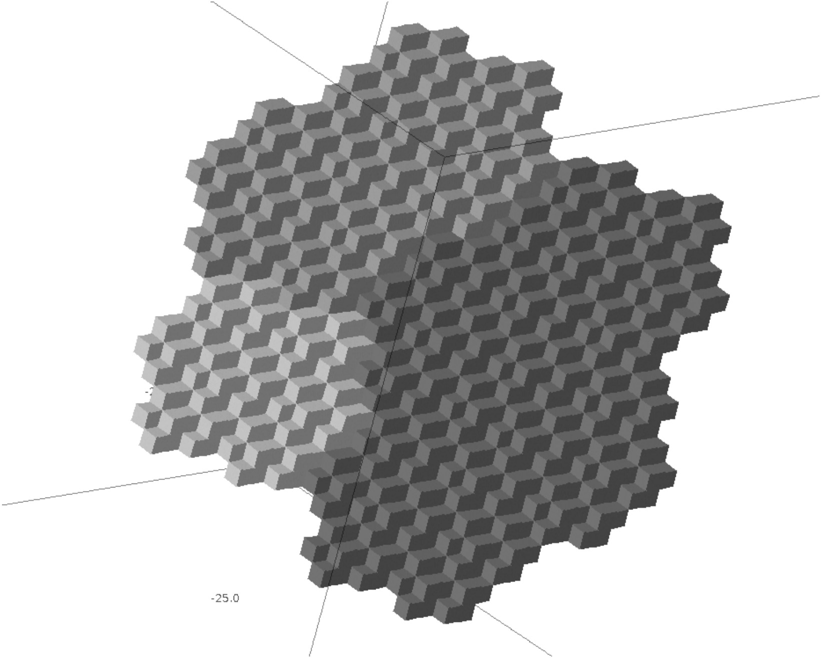
Images don't overlap [Arnoux-Ito '01]

$[\mathbf{x}, i]^* \neq [\mathbf{y}, j]^* \in$ discrete plane

$$\implies \quad \mathbf{E}_1^*(\sigma)([\mathbf{x}, i]^*) \ \cap\ \mathbf{E}_1^*(\sigma)([\mathbf{y}, j]^*) \ = \ \varnothing$$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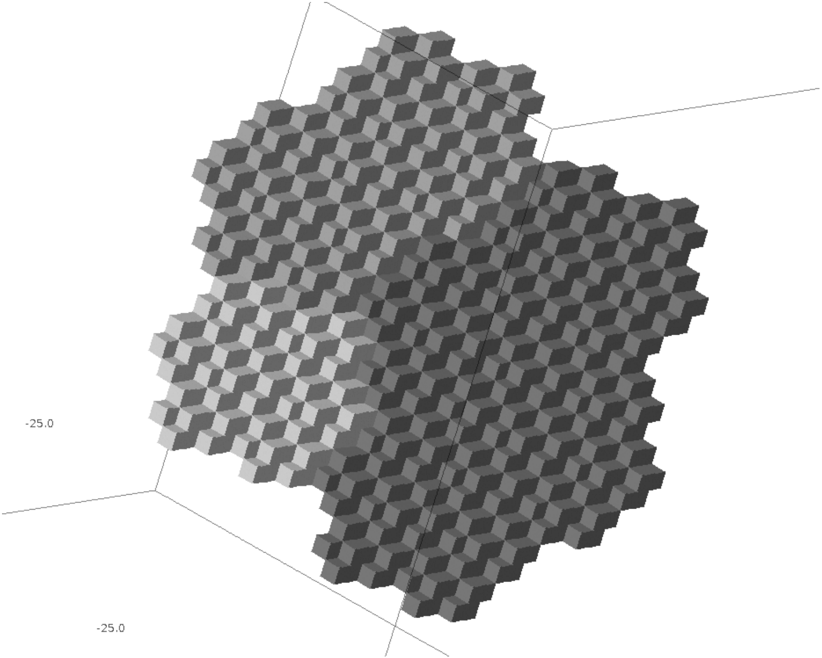
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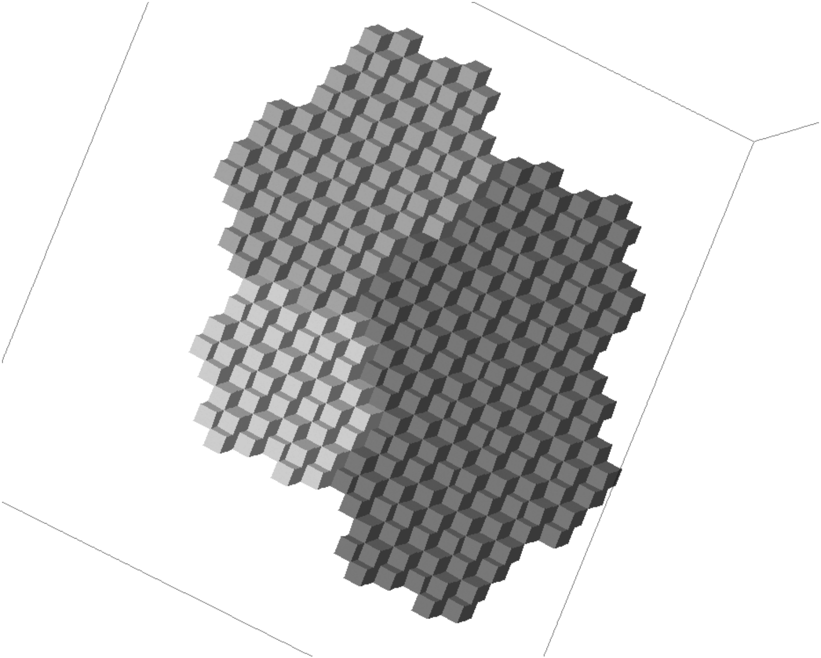

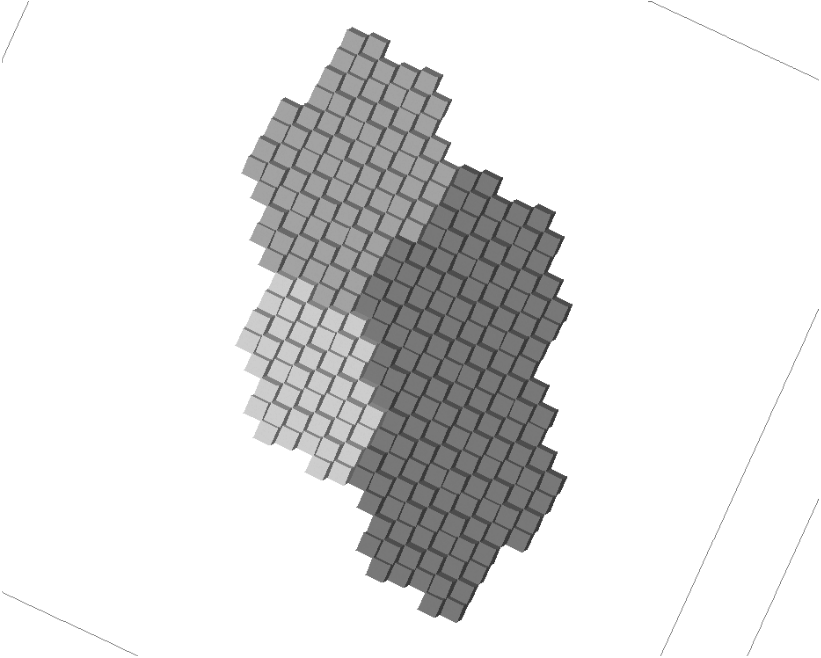

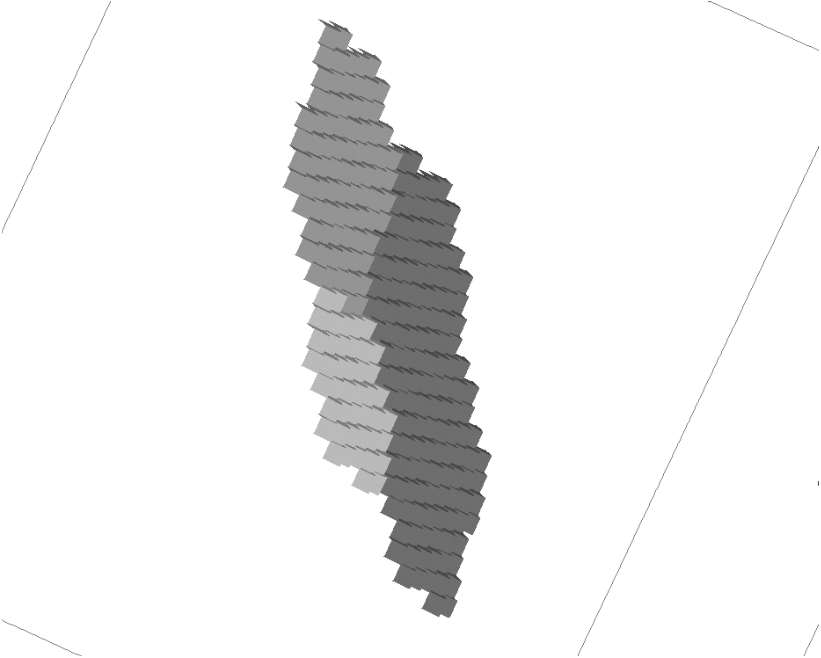

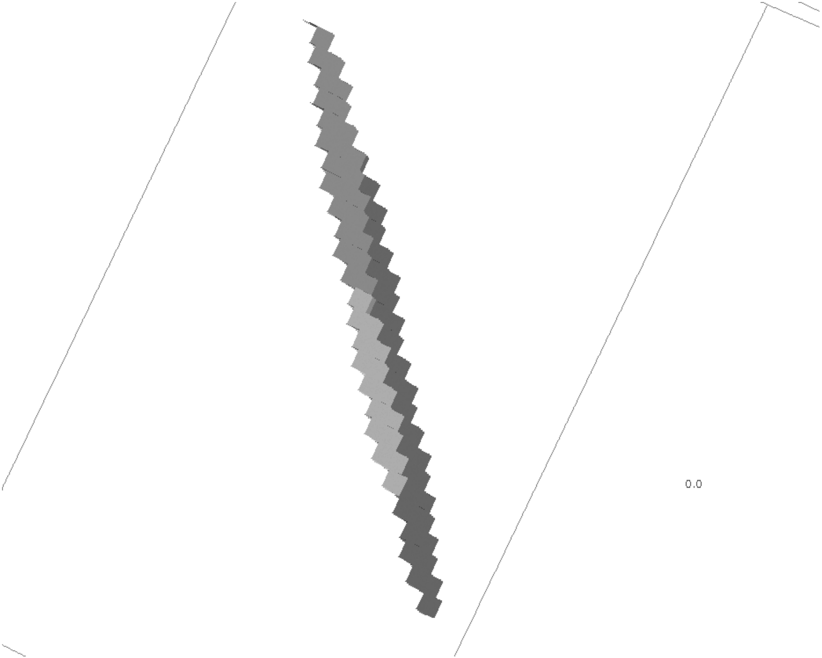
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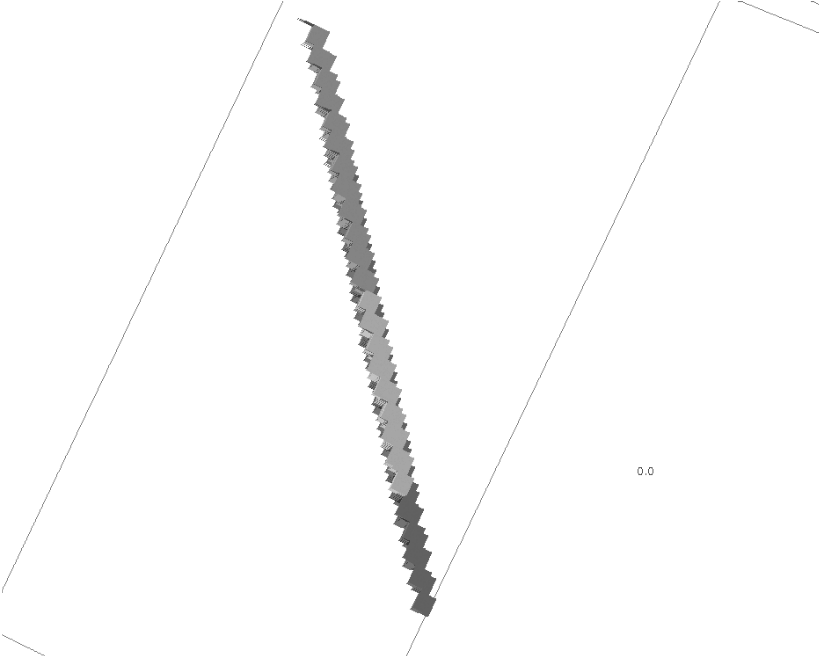
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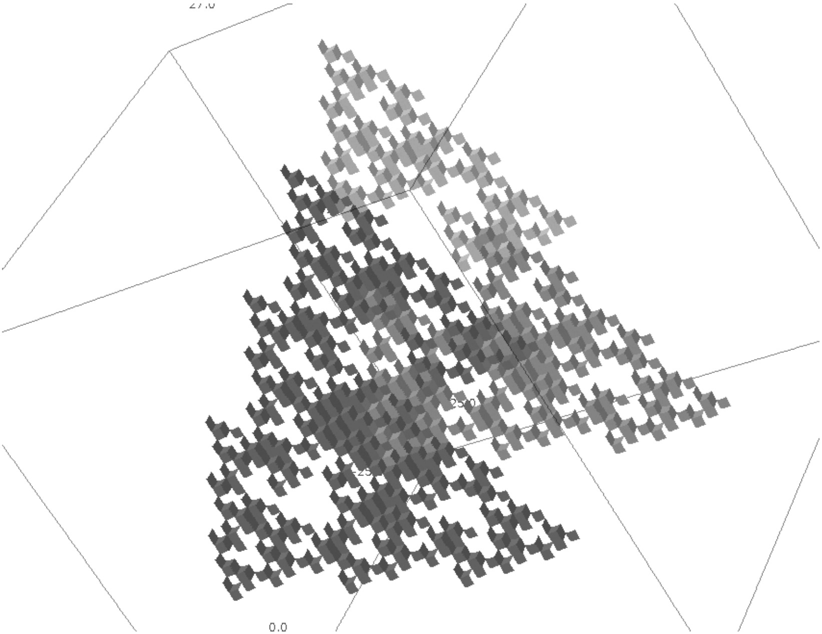
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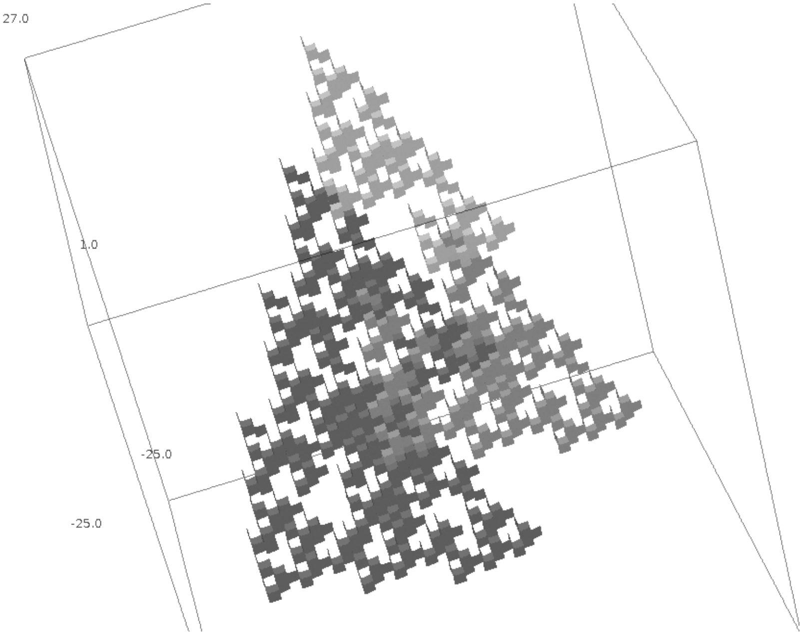
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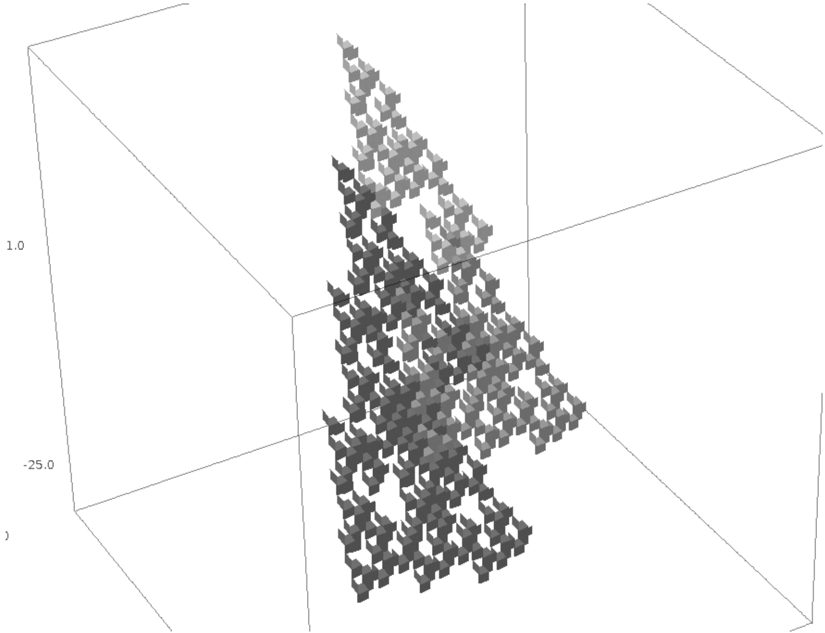
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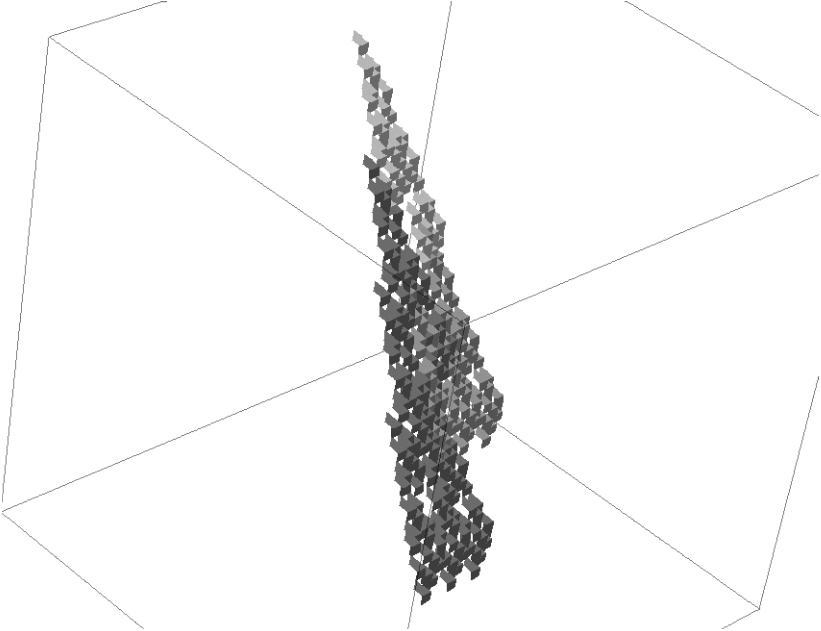

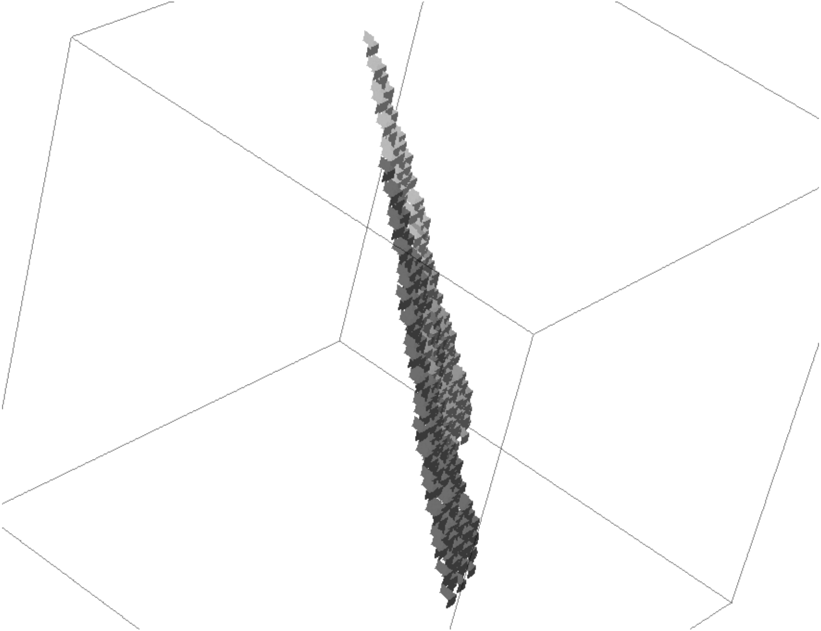

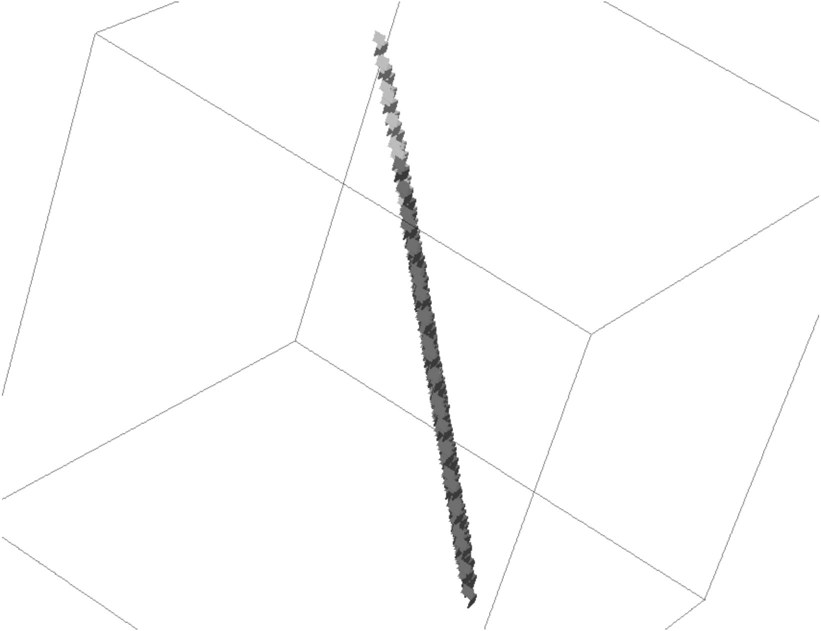

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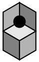

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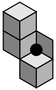

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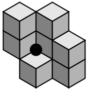

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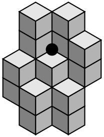

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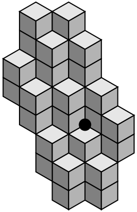

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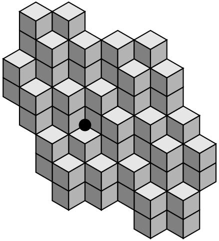

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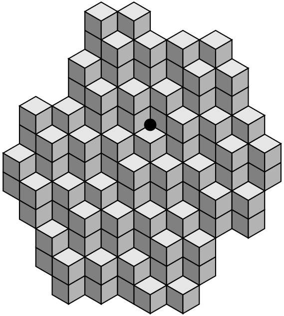

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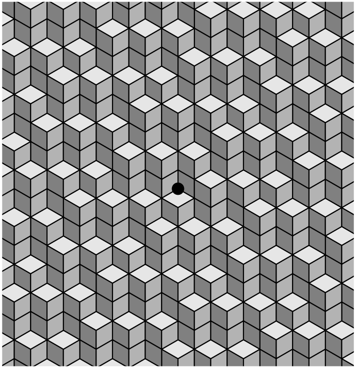
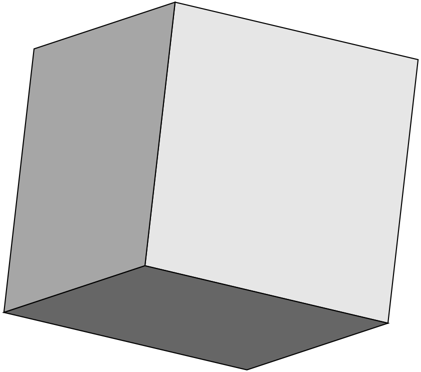


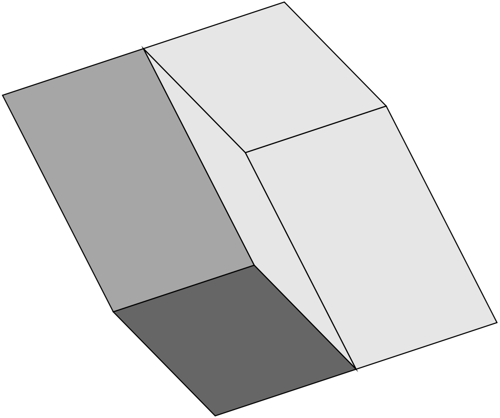
The same, with renormalization

$\pi($  $)$



The same, with renormalization

$\mathbf{M}_\sigma \pi(\mathbf{E}_1^*(\sigma)($  $))$



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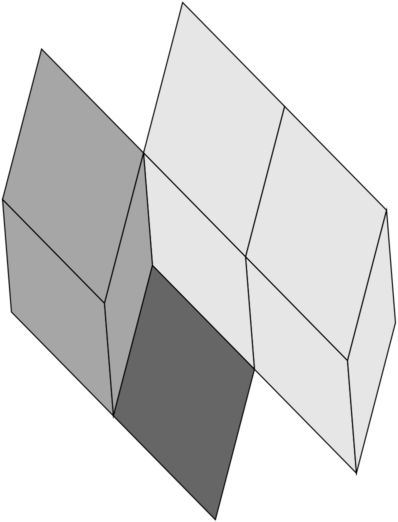
$$\mathbf{M}_\sigma^2 \pi(\mathbf{E}_1^*(\sigma)^2(\ \text{[cube]}\ ))$$



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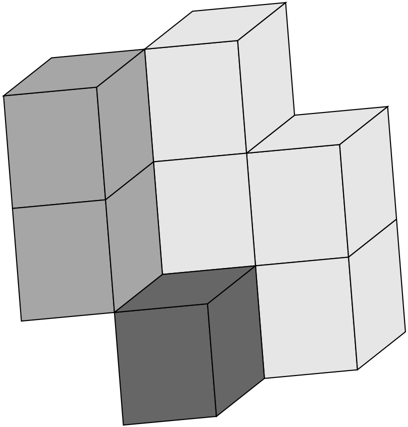

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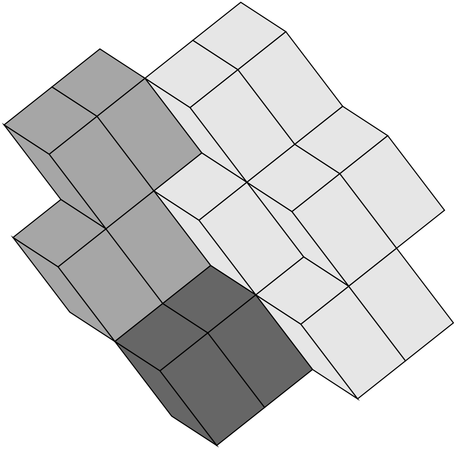

The same, with renormalization

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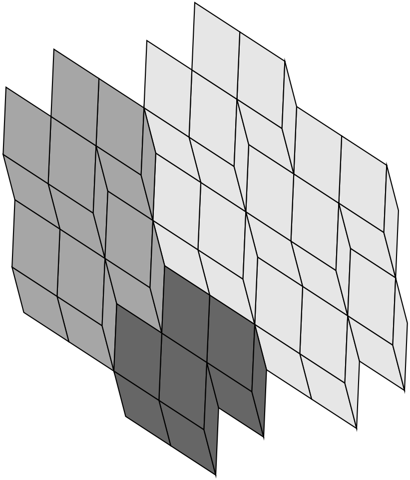
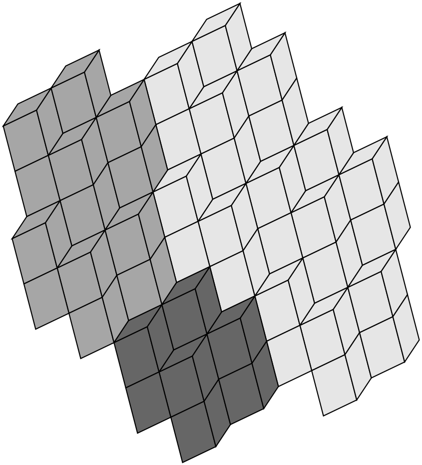


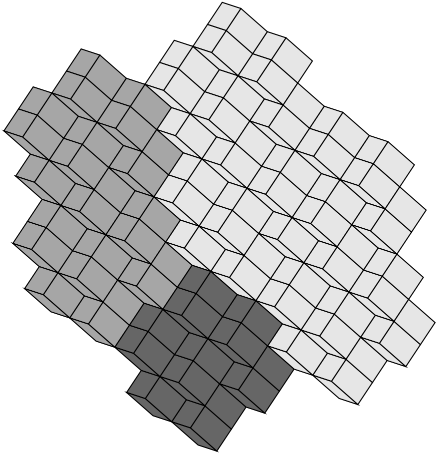
The same, with renormalization

$\mathbf{M}_\sigma^6 \pi(\mathbf{E}_1^*(\sigma)^6($  $))$



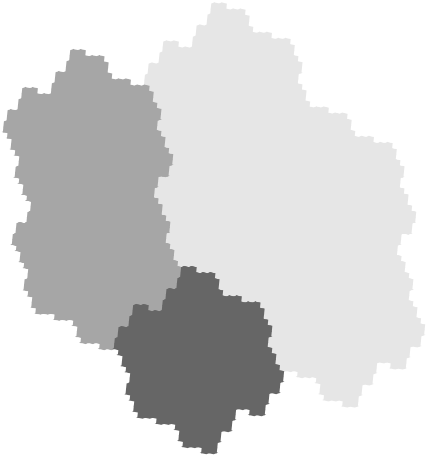
The same, with renormalization

$\mathbf{M}_\sigma^7\pi(\mathbf{E}_1^*(\sigma)^7($  $))$



The same, with renormalization

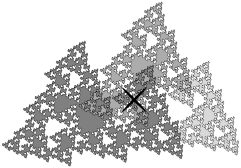
$\mathbf{M}_\sigma^\infty \pi(\mathbf{E}_1^*(\sigma)^\infty($  $))$



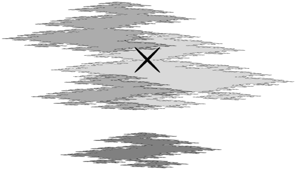
Definition [Rauzy '82, Arnoux-Ito '01]

Let $\sigma : \{1,2,3\}^* \to \{1,2,3\}^*$ be a unimodular Pisot irreducible substitution.

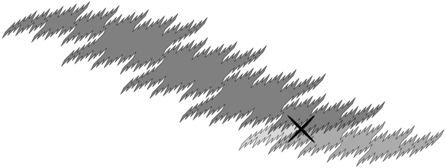
The **Rauzy fractal** associated with σ is the set $\lim_{n\to\infty} \mathbf{M}_\sigma^n \pi(\mathbf{E}_1^*(\sigma)^n($  $))$.



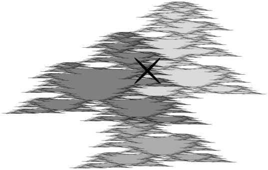
$1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$



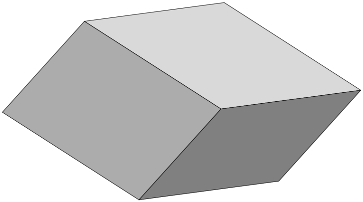
$1 \mapsto 3, 2 \mapsto 23, 3 \mapsto 31223$

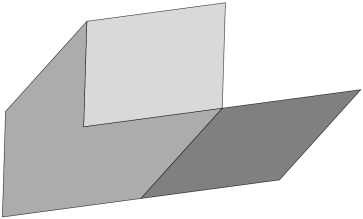


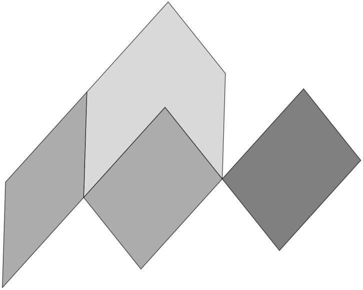
$1 \mapsto 21111, 2 \mapsto 31111, 3 \mapsto 1$

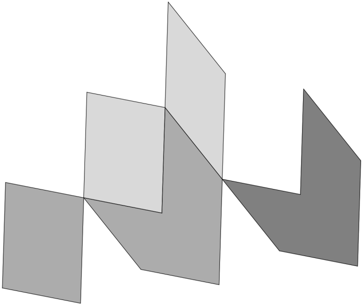


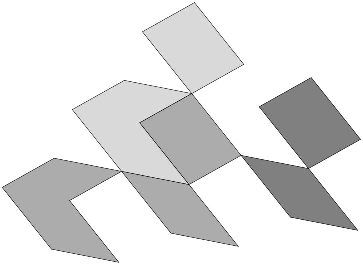
$1 \mapsto 123, 2 \mapsto 1, 3 \mapsto 31$

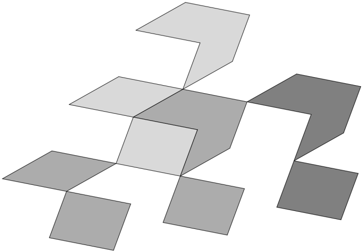


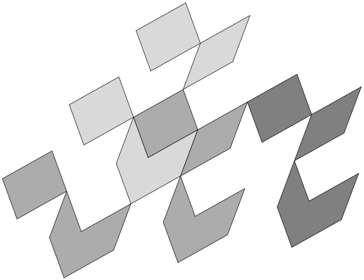


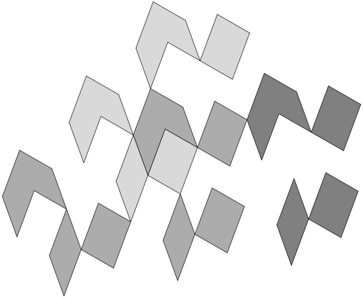


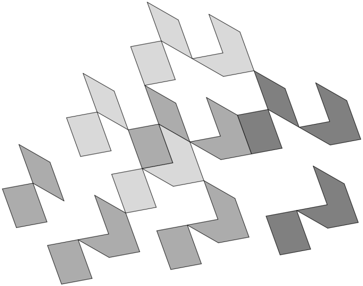


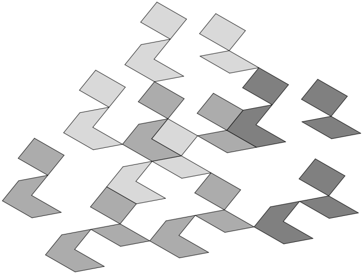


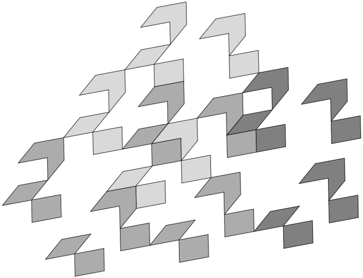


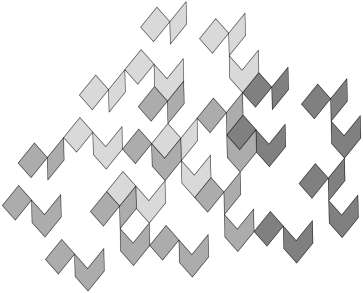


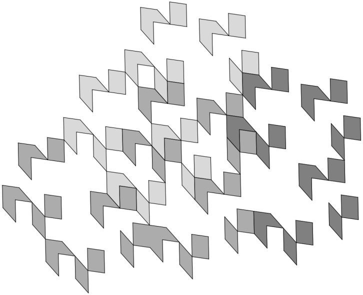


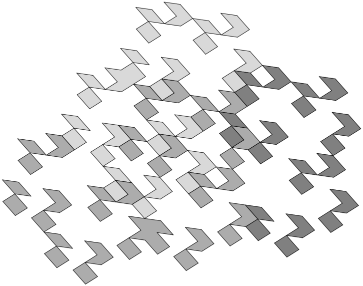


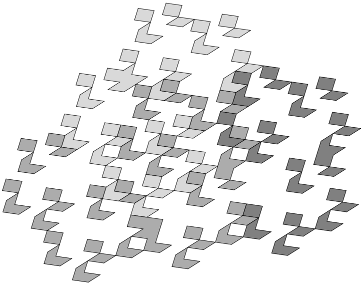


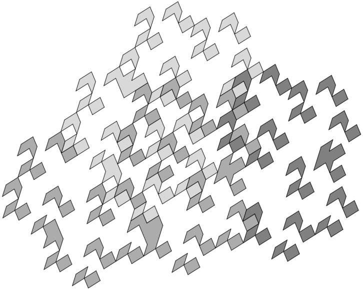


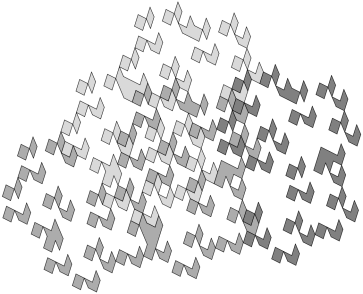


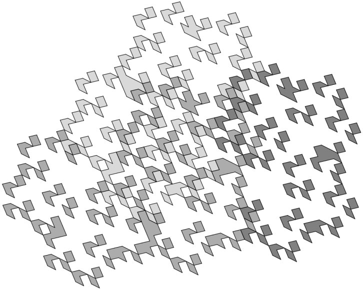


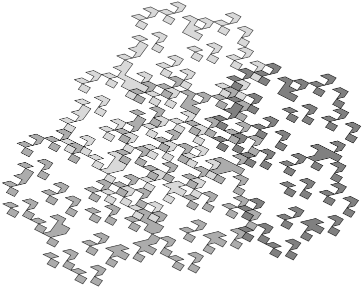


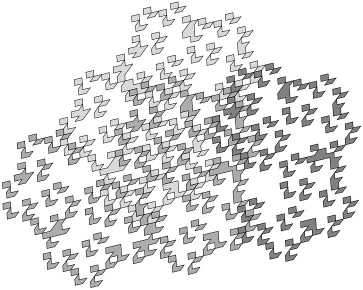


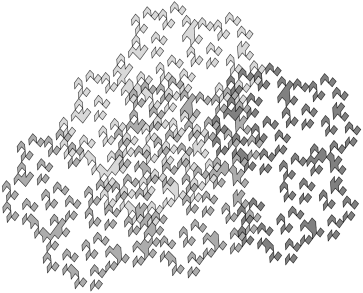


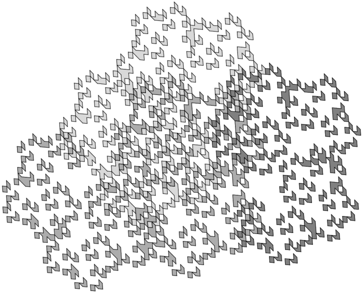


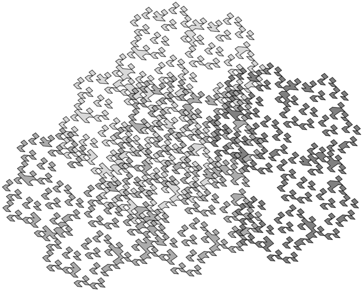


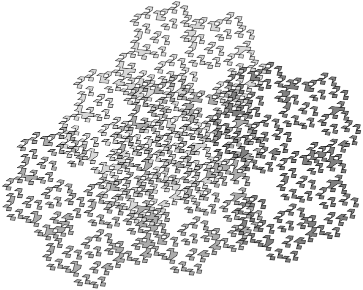












Concatenation rules

Let P be a union of unit faces.
We want to compute $\mathbf{E}_1^*(\sigma)(P)$ **without computing the new position $\mathrm{M}_\sigma^{-1}\mathbf{x}$** for every face $[\mathbf{x}, i]^* \in P$, but by **concatenating the images of the faces**.

➥ We need an analogue of $\sigma(uv) = \sigma(u)\sigma(v)$ in higher dimensions.

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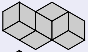
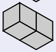
Example for $\sigma : 1 \mapsto 121312, 2 \mapsto 1312, 3 \mapsto 1121312$

We have

$\mathbf{E}_1^*(\sigma)($  $)$ = 

$\mathbf{E}_1^*(\sigma)($  $)$ =  .

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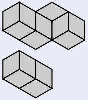 .

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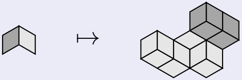
We can give a **concatenation rule**:

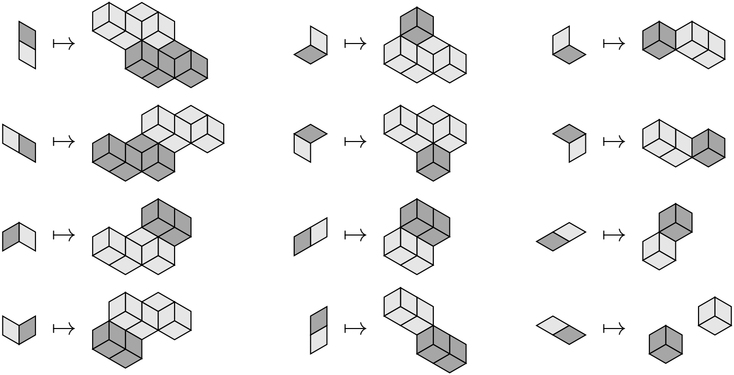


This rule must agree with $\mathbf{E}_1^*(\sigma)$.

Example (continued)

In some cases, we can describe $\mathbf{E}_1^*(\sigma)$ by a **finite set of concatenation rules** (when we iterate from ).

This is the case for our example:



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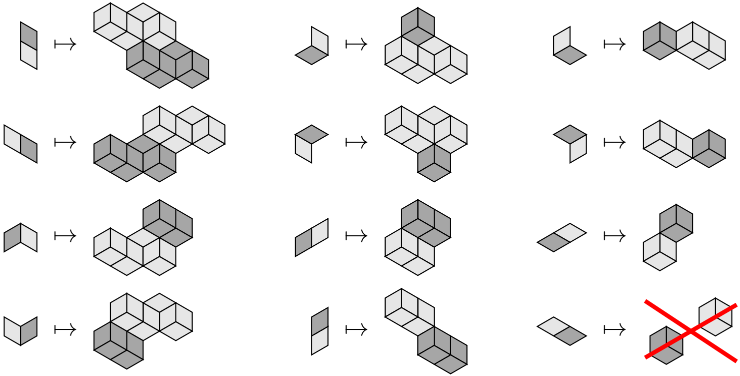

Example (continued^2)

Let's apply these rules:

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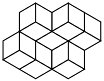

Example (continued^2)

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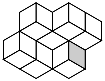
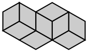
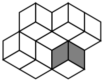


$\mapsto$

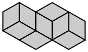


Example (continued^2)

Let's apply these rules:

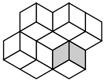


$\mapsto$

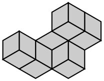


Example (continued^2)

Let's apply these rules:

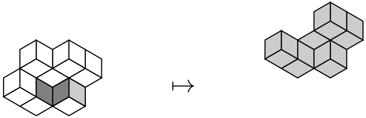


$\mapsto$



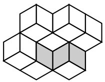
Example (continued^2)

Let's apply these rules:



Example (continued^2)

Let's apply these rules:



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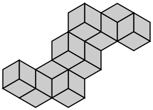
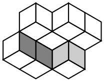


Example (continued^2)

Let's apply these rules:



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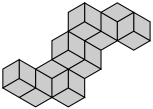

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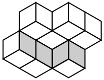

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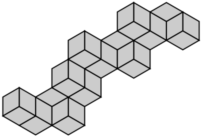

Example (continued^2)

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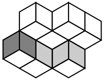
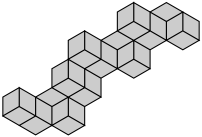


$\mapsto$



Example (continued^2)

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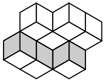

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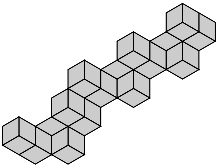

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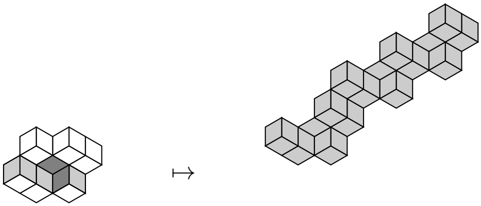

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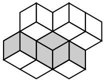

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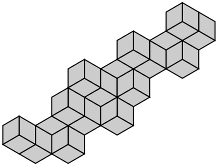

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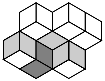

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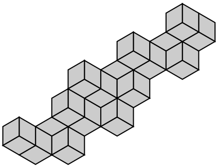

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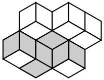

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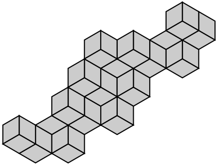

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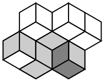

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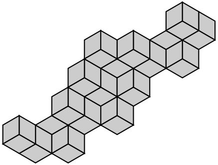

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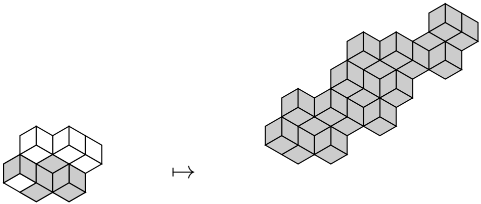

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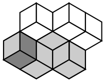

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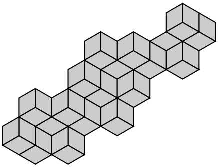

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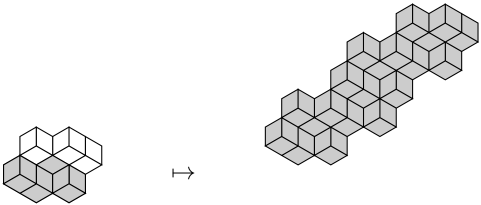

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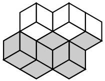

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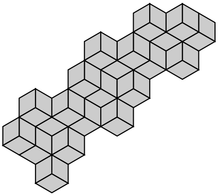

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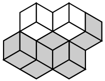
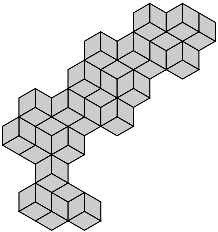


$\mapsto$



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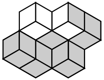
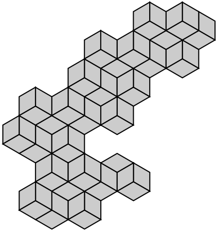


$\mapsto$



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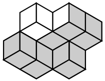

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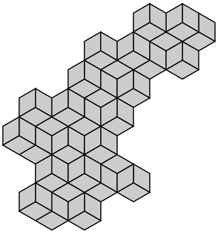

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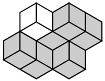

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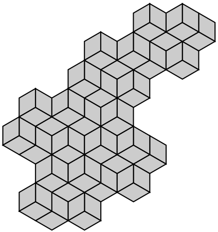

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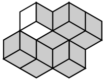

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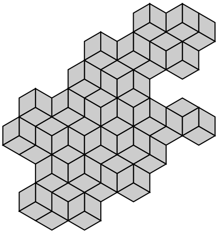

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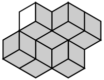

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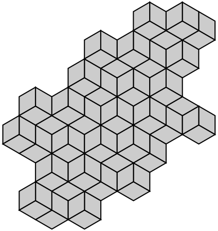

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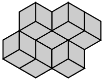
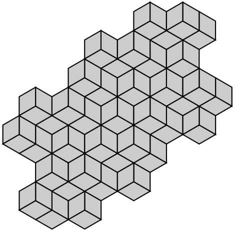


$\mapsto$



Example (continued^3)

This is an example of a **stable** set of rules (we can iterate them).

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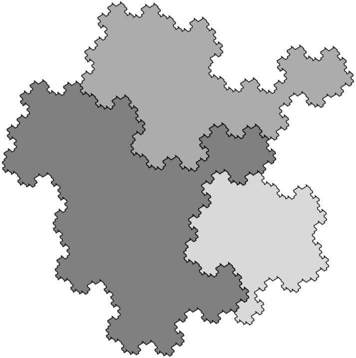
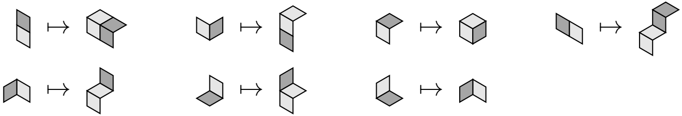
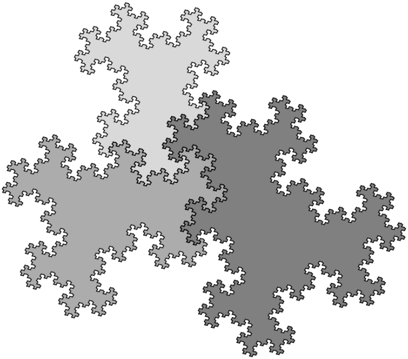
➥ The associated Rauzy fractal is **connected**, because connectedness is preserved by the Hausdorff limit (our sets are compact).



The same way, we can prove the connectedness of the fractal associated with:

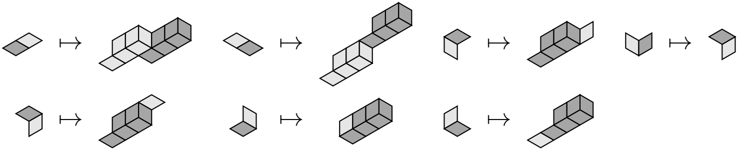
- $1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 1$:

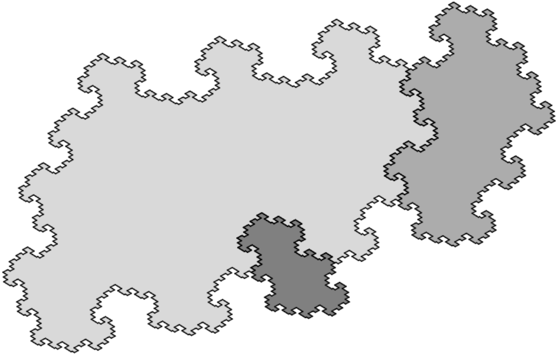




The same way, we can prove the connectedness of the fractal associated with:

- $1 \mapsto 3, 2 \mapsto 13^2, 1 \mapsto 23^3$:





Let's try do deal with some families of substitutions (not only one).

Arnoux-Rauzy substitutions

$$\sigma_1 : \begin{cases} 1 & \mapsto & 1 \\ 2 & \mapsto & 21 \\ 3 & \mapsto & 31 \end{cases} \qquad \sigma_2 : \begin{cases} 1 & \mapsto & 12 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 32 \end{cases} \qquad \sigma_3 : \begin{cases} 1 & \mapsto & 13 \\ 2 & \mapsto & 23 \\ 3 & \mapsto & 3 \end{cases}$$

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$$\mathcal{U}$$

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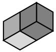
$$\mathbf{E}_1^*(\sigma_1)(\mathcal{U})$$



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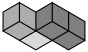
$$\mathbf{E}_1^*(\sigma_1)\mathbf{E}_1^*(\sigma_2)(\mathcal{U})$$



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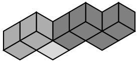

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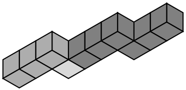

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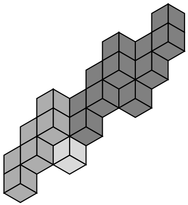

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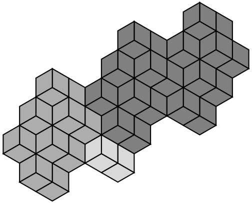

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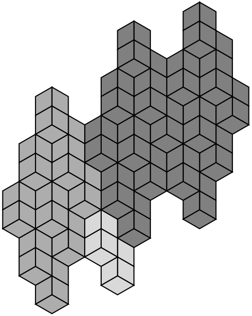
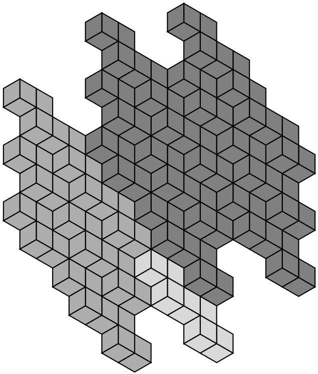


Arnoux-Rauzy substitutions

$$\sigma_1 : \begin{cases} 1 & \mapsto & 1 \\ 2 & \mapsto & 21 \\ 3 & \mapsto & 31 \end{cases} \qquad \sigma_2 : \begin{cases} 1 & \mapsto & 12 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 32 \end{cases} \qquad \sigma_3 : \begin{cases} 1 & \mapsto & 13 \\ 2 & \mapsto & 23 \\ 3 & \mapsto & 3 \end{cases}$$

$$\mathbf{E}_1^*(\sigma_1)\mathbf{E}_1^*(\sigma_2)\mathbf{E}_1^*(\sigma_1)\mathbf{E}_1^*(\sigma_1)\mathbf{E}_1^*(\sigma_3)\mathbf{E}_1^*(\sigma_2)\mathbf{E}_1^*(\sigma_3)\mathbf{E}_1^*(\sigma_2)(\mathcal{U})$$



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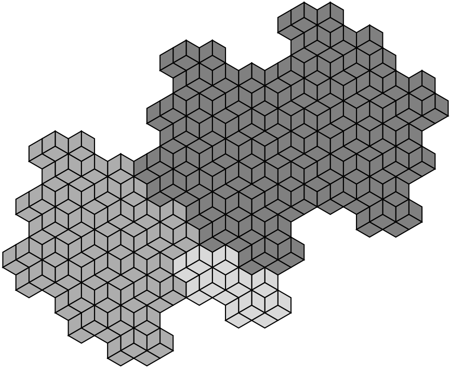
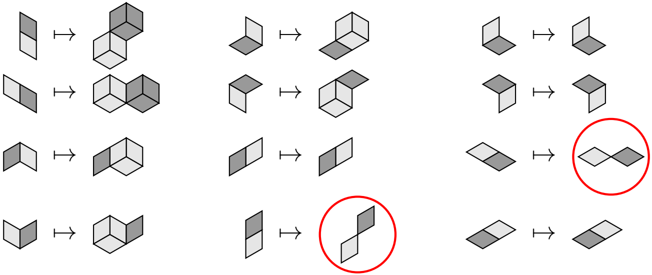
$$\mathbf{E}_1^*(\sigma_1)\mathbf{E}_1^*(\sigma_2)\mathbf{E}_1^*(\sigma_1)\mathbf{E}_1^*(\sigma_1)\mathbf{E}_1^*(\sigma_3)\mathbf{E}_1^*(\sigma_2)\mathbf{E}_1^*(\sigma_3)\mathbf{E}_1^*(\sigma_2)\mathbf{E}_1^*(\sigma_1)(\mathcal{U})$$



Connectedness of Arnoux-Rauzy fractals

Let's look at $\mathbf{E}_1^*(\sigma_1)$:



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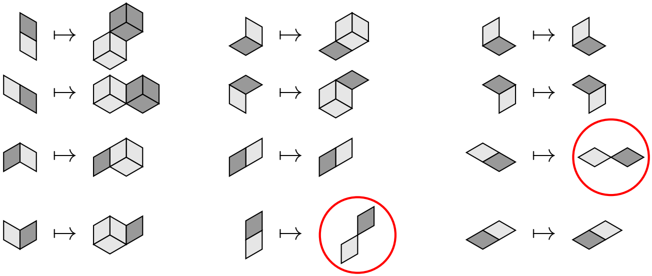
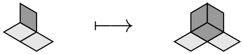


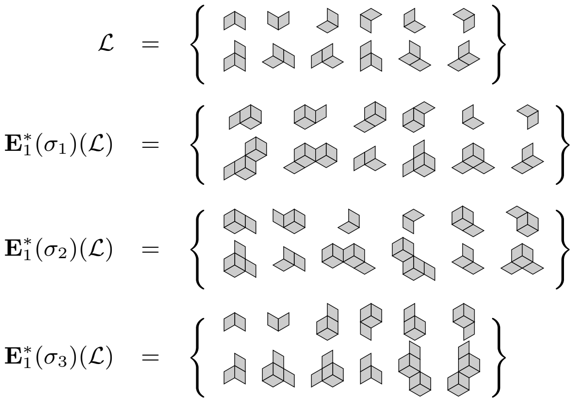
Idea: Consider larger starting patterns:



Connectedness of Arnoux-Rauzy fractals

These larger patterns yield 3 sets of 12 rules (one for each σ_i) such that:

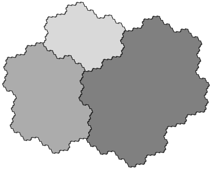
- the 3 sets of rules for σ_1, σ_2 and σ_3 are mutually stable,
- the patterns in the rules are connected.



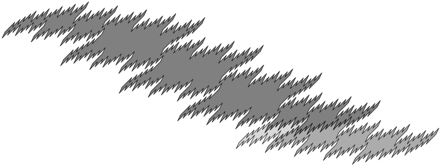
Theorem [Berthé-J-Siegel]

Let $\sigma_{i_1}, \dots, \sigma_{i_n}$ be Arnoux-Rauzy substitutions; $i_1, \dots, i_n \in \{1, 2, 3\}$. Then:

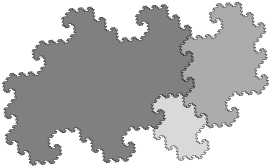
1. $\mathbf{E}_1^*(\sigma_{i_1}) \cdots \mathbf{E}_1^*(\sigma_{i_n})($  $)$ is connected
2. The fractal associated with σ is connected.



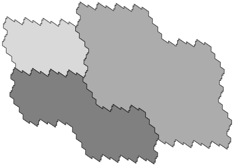
$\sigma_1\sigma_2\sigma_3$



$\sigma_1\sigma_1\sigma_1\sigma_1\sigma_2\sigma_2\sigma_2\sigma_2\sigma_3\sigma_3\sigma_3\sigma_3$



$\sigma_1\sigma_1\sigma_2\sigma_2\sigma_3\sigma_3$



$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_2\sigma_2\sigma_1\sigma_3$

Future work

- Which Arnoux-Rauzy fractals are simply connected?
- Prove topological properties for some other families of fractals.
- **Decidability:** connectedness, simple connectedness, is $\mathbf{0}$ an interior point of the fractal?, ...

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Thank you for you attention.

Any questions?