3D Rauzy fractals

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With **Benoît Loridant** and **Jörg Thuswaldner**

TU Graz
2014-04-03

Substitutions

$$\sigma : \begin{cases} 1 & \mapsto & 12 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 4 \\ 4 & \mapsto & 5 \\ 5 & \mapsto & 1 \end{cases} \qquad \mathbf{M}_\sigma = \begin{pmatrix} 1&0&0&0&1 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{pmatrix}$$

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Pisot substitution:

- Dominant eigenvalue $\beta \approx 1.325$ is a Pisot number ($\beta', \beta'' \approx -0.662 \pm 0.562i$)
- Two other eigenvalues ($\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$)

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Action of M_σ on $\mathbb{R}^5$:

- **Expanding line $\mathbb{E}$**

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Action of M_σ on $\mathbb{R}^5$:

- **Expanding line** $\mathbb{E}$
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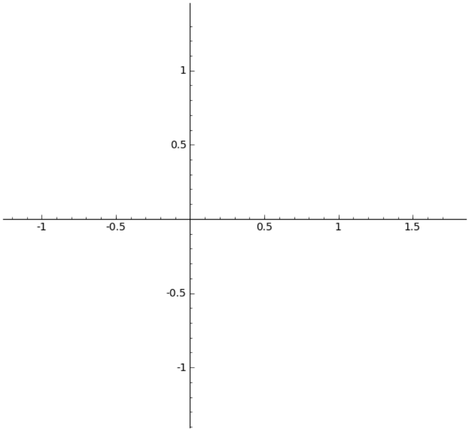
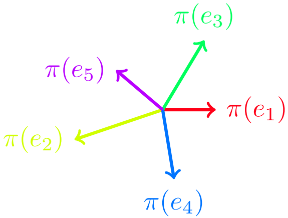
- (Supplementary space $\mathbb{H}$)

Rauzy fractal of $1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 1$

$123451121231234123451234511 23$



π = projection from $\mathbb{R}^5$ to $\mathbb{P}$ along $\mathbb{E} \oplus \mathbb{H}$



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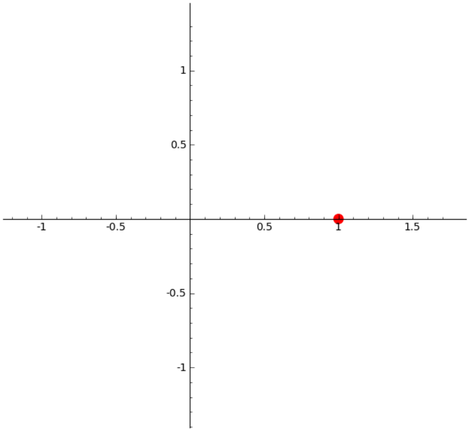

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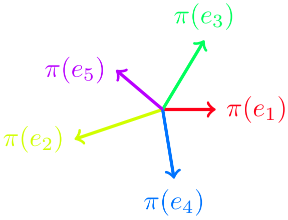

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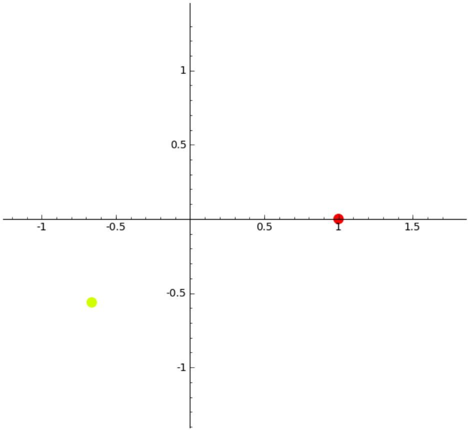
$\pi(e_1) + \pi(e_2)$



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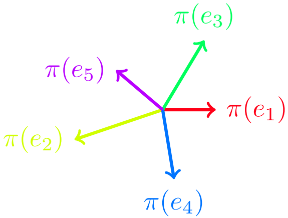

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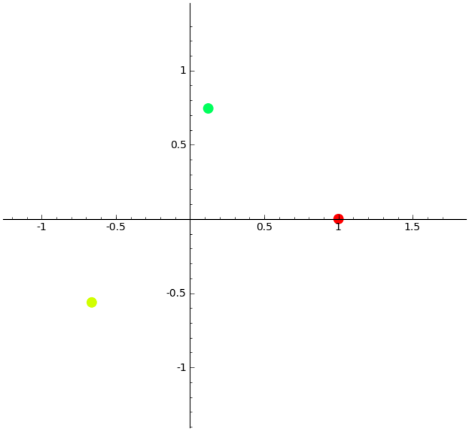

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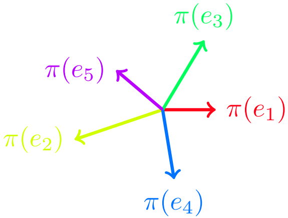

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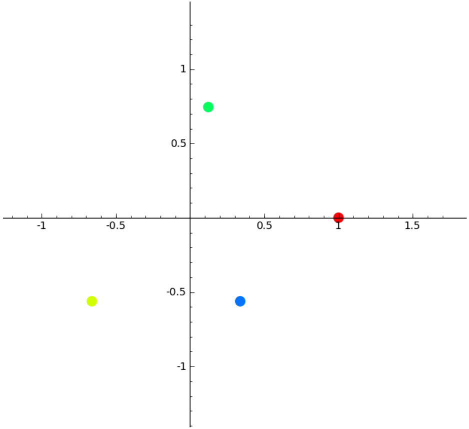

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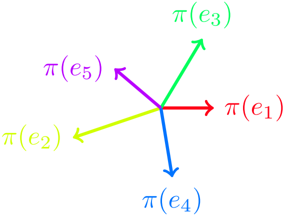

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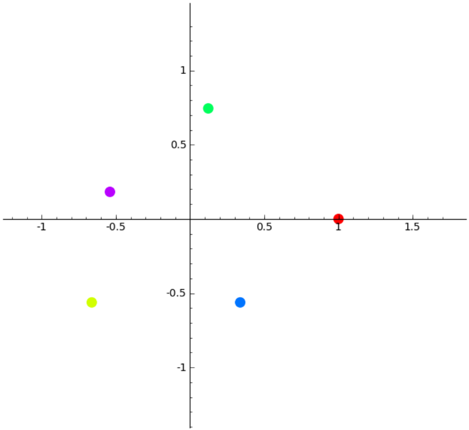

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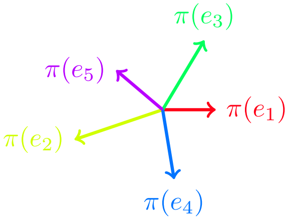

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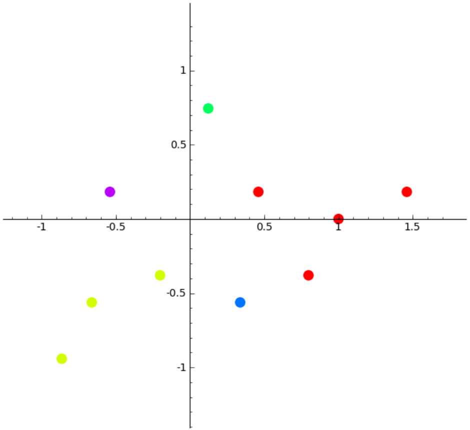

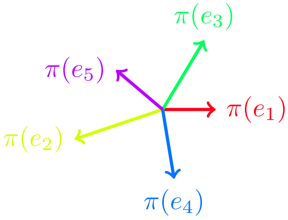

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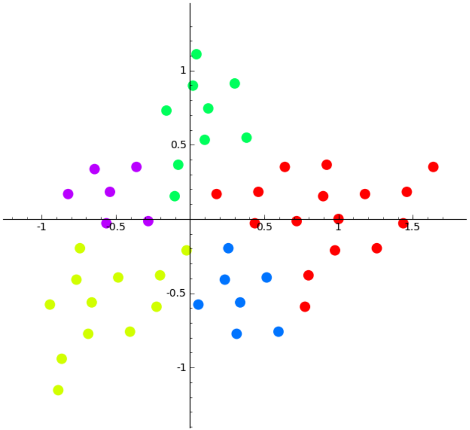

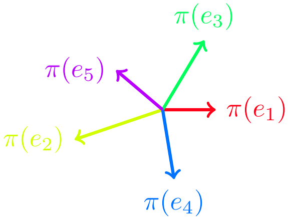

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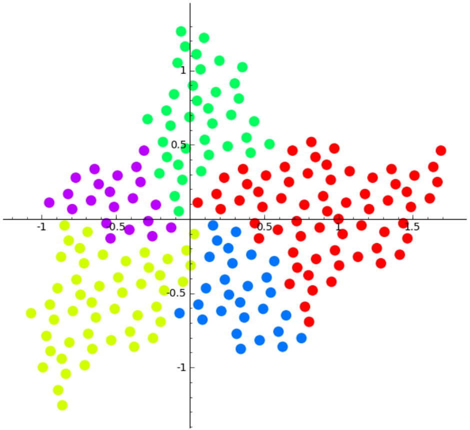

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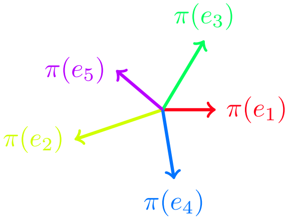

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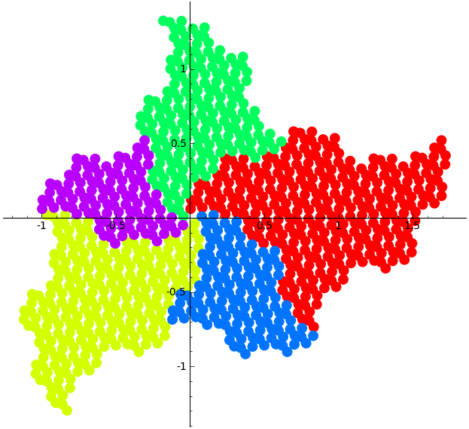

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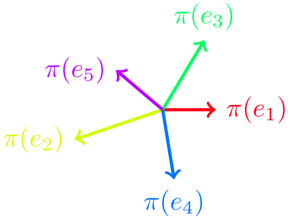

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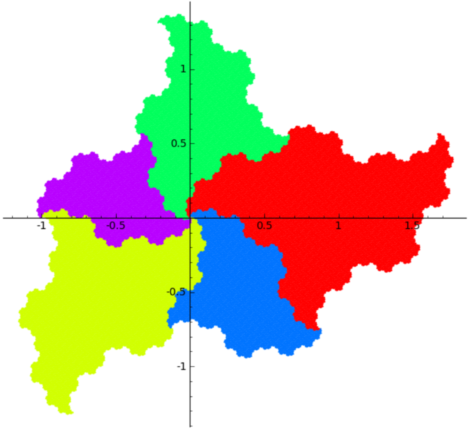

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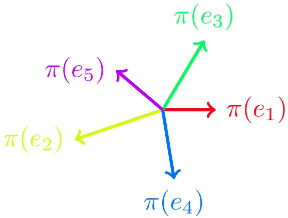

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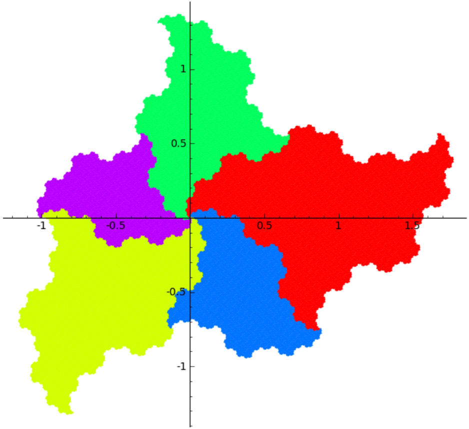

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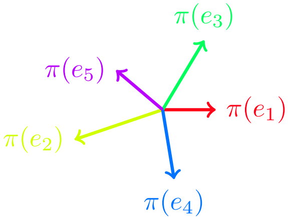
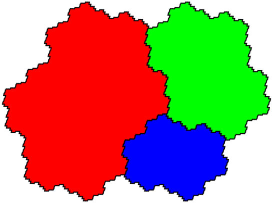


- #tiles = #letters
- dimension $= \deg(\beta) - 1$

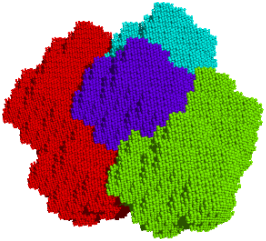
Examples

Fibonacci $1 \mapsto 12, 2 \mapsto 1$:  $\deg(\beta) = 2$

Tribonacci $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$: $\deg(\beta) = 3$



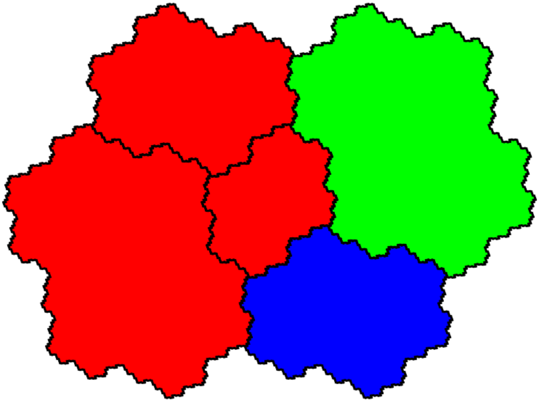
Quadribonacci $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 14, 4 \mapsto 1$: $\deg(\beta) = 4$



Iterated function system

$\mathbf{h}$ = contraction on $\mathbb{P}$ induced by $\mathbf{M}_\sigma$

$$\begin{cases} X_1 = \mathbf{h}(X_1) \cup \mathbf{h}(X_2) \cup \mathbf{h}(X_3) \\ X_2 = \mathbf{h}(X_1) + \pi(\mathbf{e}_1) \\ X_3 = \mathbf{h}(X_2) + \pi(\mathbf{e}_1) \end{cases}$$



Facts

- Solution of a graph-IFS
- Compact, nonempty interior
- Fractal boundary

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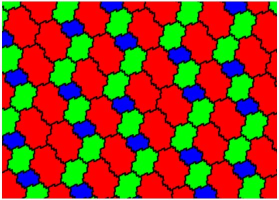

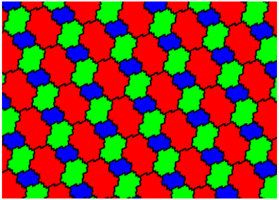

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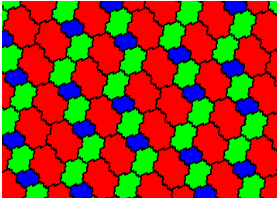

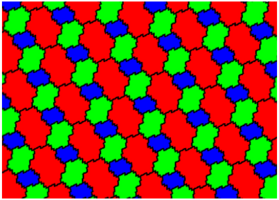

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  - dynamics of the subshift generated by σ
  - β-numeration (β-tiles)
  - ...

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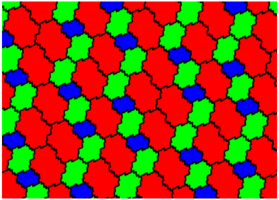

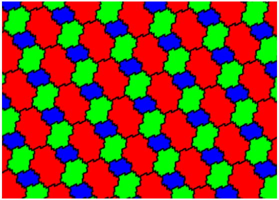

- Links with:
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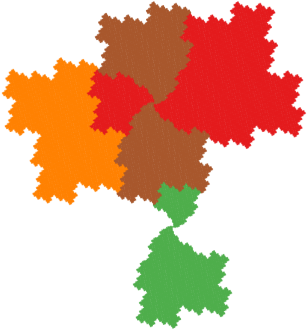
  - ...
- Rich **topological properties**

Examples

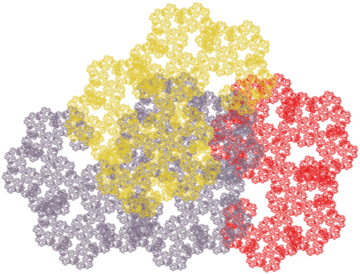
- Homeomorphic to a disc

Examples



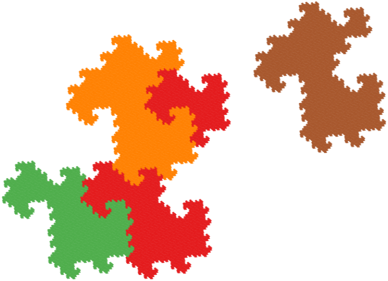
- Not homeomorphic to a disc
- Connected, simply connected

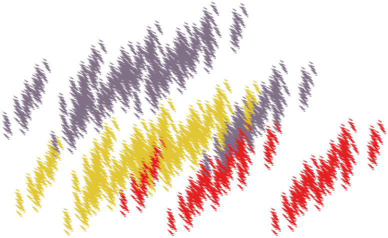
Examples



- Uncountable fundamental group
- Connected

Examples



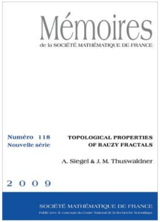


- Not connected
- Can be quite ugly

2D

[Siegel-Thuswaldner 2009]: **algorithms** for:

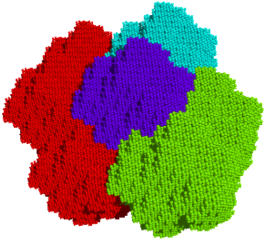
- connectedness
- homeomorphy to a disc
- uncountable fundamental group
- tiling properties
- k-multiple points



Works in 2D only (heavily relies on planar topology)

3D

Quadribonacci: **is this a 3-ball?**



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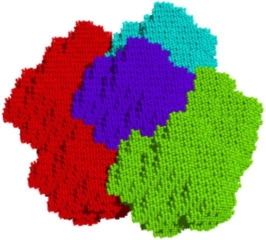

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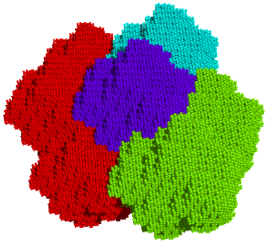

Difficulties:

- boundary is a ~~curve~~
- Jordan curve theorem false in 3D anyway. . .

3D

Simpler classes of fractals: same problems

3D

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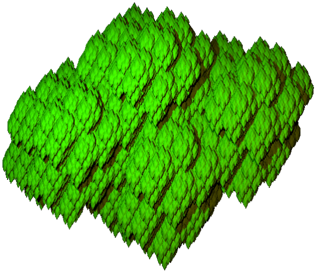
Geldbrich's twin dragon:

$$X = AX \cup \left(AX + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

with $A = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$



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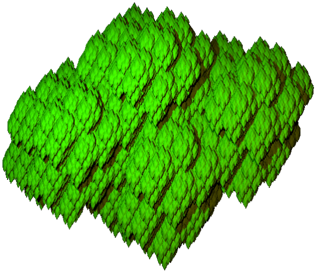

Recent result [Conner-Thuswaldner 2014]: X is a 3-ball!

3D

Further work: adapt [Conner-Thuswaldner] for Rauzy fractals

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Crucial tool: boundary/contact graphs

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- understand how tiles intersect in the self-similar tiling of $\mathbb{R}^d$

3D

Further work: adapt [Conner-Thuswaldner] for Rauzy fractals

Crucial tool: boundary/contact graphs

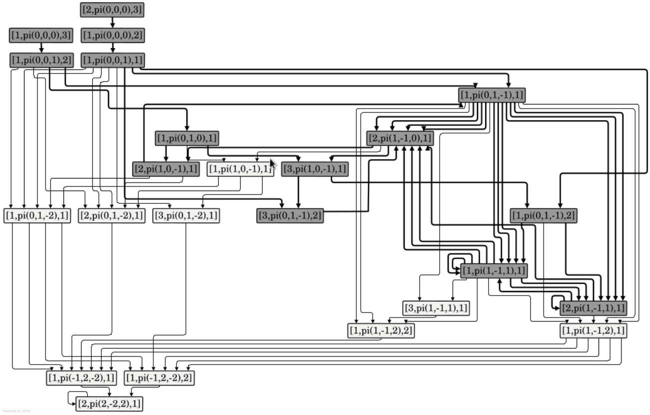
- understand how tiles intersect in the self-similar tiling of $\mathbb{R}^d$
- computable from the substitution defining the fractal

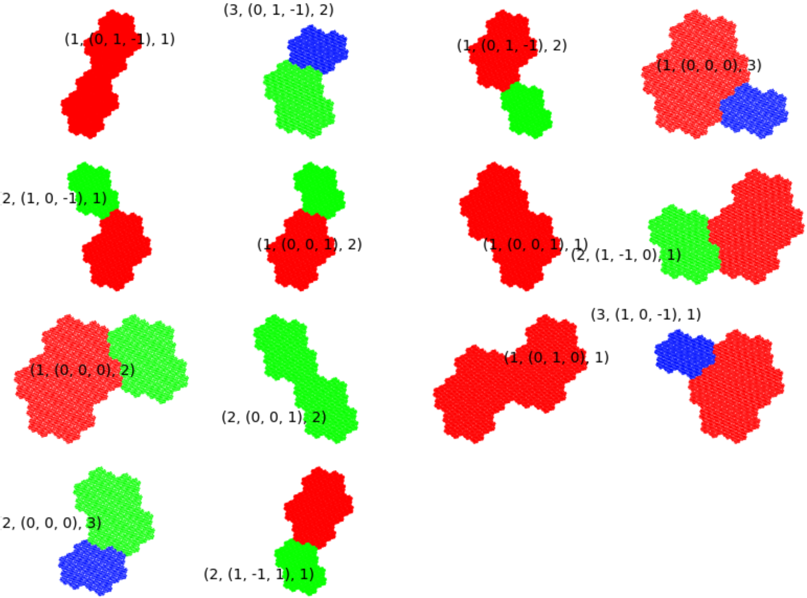
3D

Further work: adapt [Conner-Thuswaldner] for Rauzy fractals

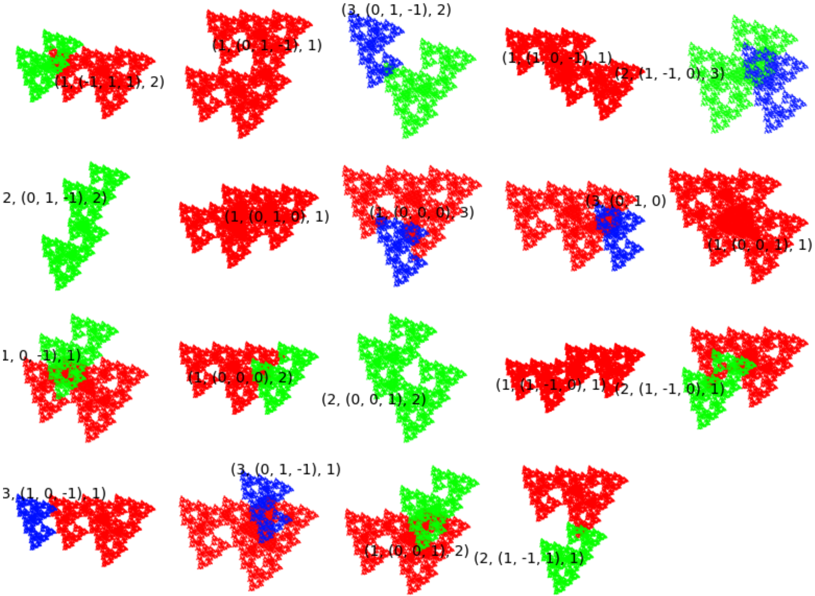
Crucial tool: boundary/contact graphs

- understand how tiles intersect in the self-similar tiling of $\mathbb{R}^d$
- computable from the substitution defining the fractal





(1, (0, 1, -1), 1)
(3, (0, 1, -1), 2)
(1, (0, 1, -1), 2)
(1, (0, 0, 0), 3)
2, (1, 0, -1), 1)
(1, (0, 0, 1), 2)
(1, (0, 0, 1), 1)
(2, (1, -1, 0), 1)
(3, (1, 0, -1), 1)
(1, (0, 0, 0), 2)
(2, (0, 0, 1), 2)
(1, (0, 1, 0), 1)
2, (0, 0, 0), 3)
(2, (1, -1, 1), 1)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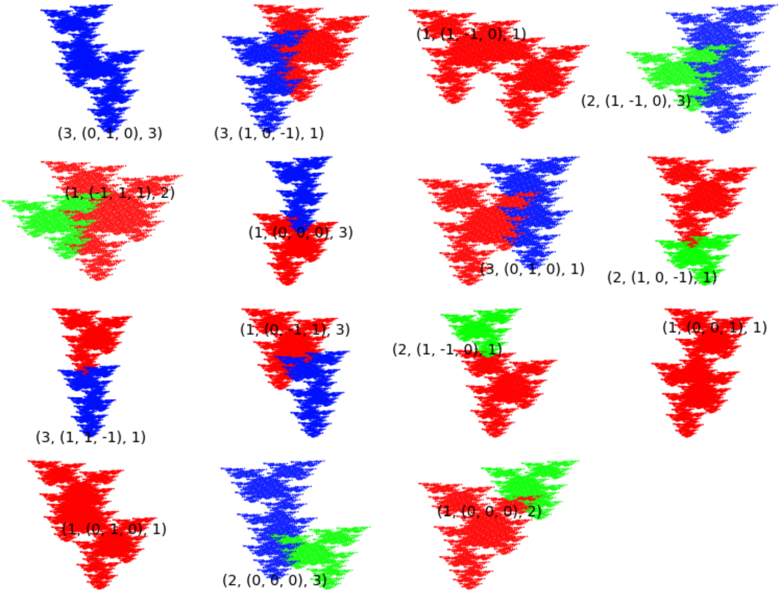
(1, (0, 1, -1), 1)
(3, (0, 1, -1), 2)
(1, (1, 0, -1), 1)
(2, (1, -1, 0), 3)
2, (0, 1, -1), 2)
(1, (0, 1, 0), 1)
(1, (0, 0, 0), 3)
(3, (0, 1, 0)
(1, (0, 0, 1), 1)
1, 0, -1), 1)
(1, (0, 0, 0), 2)
(2, (0, 0, 1), 2)
(1, (1, -1, 0), 1)
(2, (1, -1, 0), 1)
3, (1, 0, -1), 1)
(3, (0, 1, -1), 1)
(1, (0, 0, 1), 2)
(2, (1, -1, 1), 1)


(3, (0, 1, 0), 3)
(3, (1, 0, -1), 1)
(1, (1, -1, 0), 1)
(2, (1, -1, 0), 3)
(1, (-1, 1, 1), 2)
(1, (0, 0, 0), 3)
(3, (0, 1, 0), 1)
(2, (1, 0, -1), 1)
(3, (1, 1, -1), 1)
(1, (0, -1, 1), 3)
(2, (1, -1, 0), 1)
(1, (0, 0, 1), 1)
(1, (0, 1, 0), 1)
(2, (0, 0, 0), 3)
(1, (0, 0, 0), 2)

Thank you for you attention