Tiling the plane with one tile

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1 Introduction

Tiling the plane is a very old activity of mankind. In the book of B. Grunbaum and G.C. Shephard, Tilings and patterns [3], one can find many beautiful examples of decorative tilings belonging to nearly all ancient civilizations. Most decorative tilings are those we call *periodic*, this meaning that they are invariant by two independant translations.

Among periodic tilings some are called below *regular*: if the tiling uses just one tile and tranlated instances of itself, the tiling is said to be *regular* if and only if the surrounding of each instance of the tile in the tiling is the same. Grunbaum and Shephard [3] give to regularity a completely different meaning (the tilings which are shown on page 34 of their book are not regular in their sense but clearly regular in our sense).

From our definitions it follows that a periodic tiling U by a finite number of different tiles induces a regular tiling with just one tile T: this means that there exists a tile T exactly covered by a finite number of instances of the given tiles such that U contains a regular tiling by T. This can be shown in the following example. The well known and widely used tiling with one regular octogon and one square contains the regular tiling by one tile exactly covered by one octogon and one adjacent square.

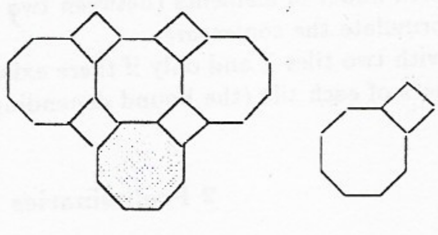

Fig. 1.1

We call below *exact* a tile such that there exists at least one tiling of the plane by translated instances of this tile. And our paper is devoted to the characterization of exact tiles.

We first investigated exact *polyominoes* (a polyomino being a polygon exactly covered by unit squares whose edges are horizontal and vertical). We refined a recent result of J.van Leeuwen and H.A.G.Wijshoff [10] about exact polyominoes. We proved that a polyomino is exact if and only if it is a *pseudo-hexagon*, and that every tiling with an exact polyomino is half-periodic (i.e. invariant by some translation). From the result of Jan van Leeuven and Wijshoff follows that it is decidable whether a polyomino is exact and this was their aim when studying skewing schemes and data transfer functions in various types of parallel processing machines. The purely geometrical properties of exact polyominoes do not appear in their work. On the contrary we focused on these properties and this lead us to extend the results on polyominoes to the widest possible family of tiles i.e. all the subsets of the plane which are homeomorphic to a closed disk and whose boundary is piecewise C^2, with a finite number of inflexion points (we do need be able to define the length of the boundary). All the results remain true, namely the fact that a tile is exact if and only if it admits a surrounding by translated images of itself (the number of which can be proved to be at least 4 and at most 8), this being equivalent to saying that the tile is a pseudo-hexagon (which can be degenerated). Such an exact tile and a tiling of the plane by this tile are shown below.

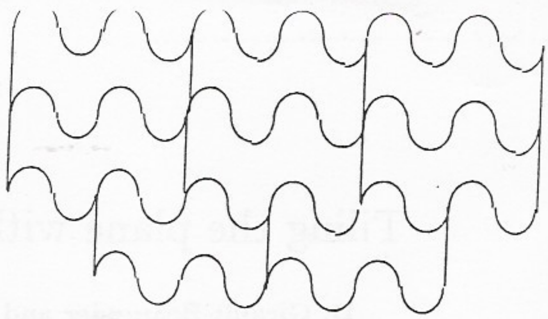

Fig. 1.2

The fact that the exact tile shown above is a pseudo-hexagon is shown by the following figure:

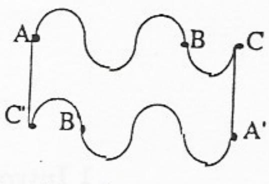

Fig. 1.3

The translation $\overrightarrow{AB'}$ maps the edge $[AB]$ onto the edge $[B'A']$, the translation $\overrightarrow{BC'}$ maps the edge [E onto the edge $[C'B']$ and the translation $\overrightarrow{CA}$ maps the edge $[CA']$ onto the edge $[AC']$.

Furthermore the two following results hold:

-there exists a tiling with one (exact) tile if and only if there exists a regular tiling

-all tilings by an exact tile are half-periodic.

A conclusion of this work is that Hao Wang's conjecture [8] holds for just one tile. Hao Wang conjectur that for all finite set of tiles there exists a tiling if and only if there exists a periodic one and this conjectu was proved to be false by exhibiting sets of tiles such that the set of tilings is not empty but does not conta a periodic one. Such sets have been exhibited in that order by R.Berger[2], M.Robinson [7] and R. Penrc [6], the last one being the simplest. In fact all these authors allow rotations of the tile but their results c be easily converted into results for tiles which are only translated (the number of possible rotations for the tiles is finite). Penrose's set of tiles (kites and darts) is equivalent to a set of tranlated tiles containing elements. And thus the problem is still open to decide whether there exists a tiling of the plane for a give set of tiles containing a small nuber of elements (between two an twenty). We may guess that the proble is decidable for two and formulate the conjecture:

-there exists a tiling with two tiles if and only if there exists an exact tile which is exactly covered by bounded number of instances of each tile (the bound depending on the lengths of the two tiles) .

2 Preliminaries

The tiles we deal with are homeomorphic to a closed disk. Their *boundary* is a curve, oriented in the clockwise sense (which will be named the *positive or direct sense*), and we assume (it is a reasonable assumption after all) that this boundary is piecewise C^2 and that each component arc of class C^2 admits a finite number of inflexion points. One consequence of this hypothesis is that the boundary has a length. Let q be a tile. Two points A and B on the boundary of q define an oriented Jordan arc which is the path from A to B in the positive sense along the boundary of q. It will be denoted by *[AB]*. The image of q in the translation of vector u will be denoted by $q(u)$, and $q(u)$ is called an *instance* of q. The boundary of a tile q is denoted by $b(q)$ and its length is $|b(q)|$. If A belongs to the boundary of q, we denote by A' the "symmetric" point of A on this boundary, i.e. the point such that $|[AA']| = |[A'A]| = |b(q)|/2$ (it implies that $(A')' = A$). Two instances $q(u)$ and $q(v)$ are said to be

- *neighbouring* if $q(u) \cap q(v)$ is a nonempty set with an empty interior

- *simply neighbouring* if $q(u)$ and $q(v)$ are neighbouring and $q(u) \cap q(v)$ is a connected set
- *adjacent* if $q(u)$ and $q(v)$ are simply neighbouring and $q(u) \cap q(v)$ is not reduced to a point.

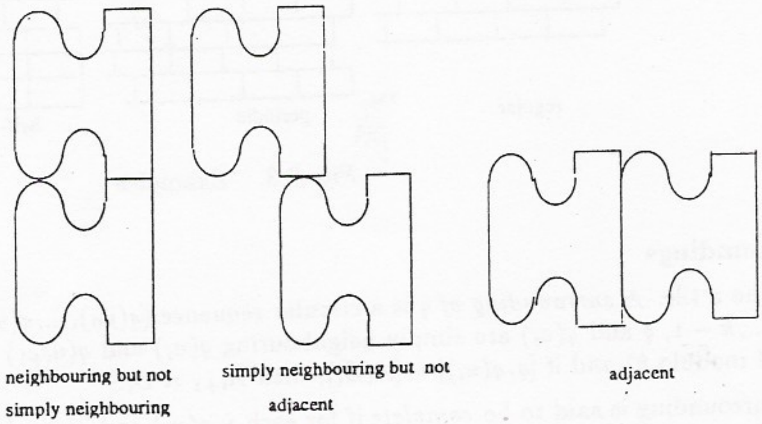

Fig. 2.1

A *tiling of the plane P by the tile q* can be represented as a set U of vectors such that $P = \bigcup_{u \in U} q(u)$ and for two distinct vectors u and v of U, $q(u)$ and $q(v)$ are not overlapping i.e. are disjoint or neighbouring.

We will focus on some objects that will play an important role later on, namely the *edges* of a tiling. The edges of a tiling are the common boundaries between two simply neighbouring tiles. We have to give a precise definition of this notion. Let $q(u)$ and $q(v)$ be two simply neighbouring tiles; let us suppose that A and B are the extremities of the arc $q(u) \cap q(v)$ such that $[BA]$ is the directly oriented arc on $q(u)$ and $[AB]$ is the directly oriented arc on $q(v)$ (Fig.2.2 a)

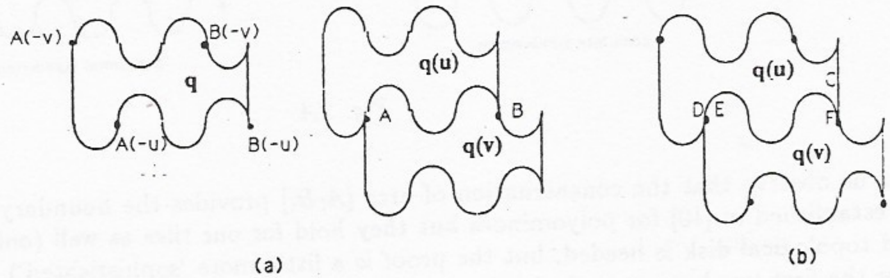

Fig. 2.2

We denote by $[q(u), q(v)]$ the edge of $q(u)$ related to $q(v)$, that is the arc $[BA]$ *referred to the reference tile q*, so it is the arc $[B(-u)A(-u)]$ of q. We define in the same way $[q(v), q(u)] = [A(-v)B(-v)]$. And, by abuse of notation, if $[q(u), q(v)] = [CD]$ and $[q(v), q(u)] = [EF]$, we will represent these edges on $q(u)$ and $q(v)$ as shown in Fig.2.2.b

A tile q is *exact* if it can tile the plane. A tiling U of the plane with an exact tile q is said to be

- *periodic* if there exist two independent vectors u and v such that $U = U + u = U + v$
- *regular* if there exist two independent vectors u and v and a vector u_0 such that

$$U = u_0 + \{ku + k'v / k, k' \in \mathbf{Z}\}$$

- *half-periodic* if there exists a vector $u \neq 0$ such that

$$U = U + u$$

(Fig.2.3)

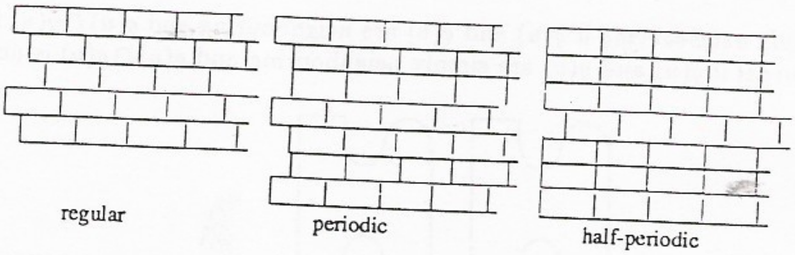

Fig. 2.3 Examples

Surroundings

Let q be a tile. A *surrounding of* q is a circular sequence $(q(u_0), ..., q(u_{k-1}))$ of instances of q such th: for $i = 0, ..., k-1$, q and $q(u_i)$ are simply neighbouring $q(u_i)$ and $q(u_{i+1})$ are simply neighbouring (indic are defined modulo k) and if $[q, q(u_i)] = [A_i B_i]$, then $A_{i+1} = B_i$.

The surrounding is said to be *complete* if for each i, $q(u_i)$ and $q(u_{i+1})$ are adjacent, *minimal* if for ea i, $|[q, q(u_i)]| > 0$ (Fig.2.4).

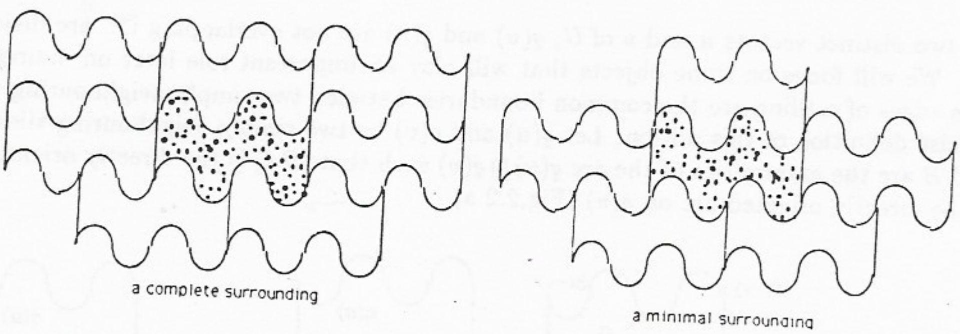

Fig. 2.4

Let us observe that the concatenation of arcs $[A_i B_i]$ provides the boundary $b(q)$. We now give some results established in [10] for polyominoes but they hold for our tiles as well (only the fact that the tile is a closed topological disk is needed; but the proof is a little more 'sophisticated'). The proof is exactly the same for the first two lemmas replacing "a cell" by "an interior point".

Lemma 2.1.— *If the tiles q and $q(u)$ don't overlap then q, $q(u)$, $q(2u)$ don't overlap each other.*(Fig 2.5)

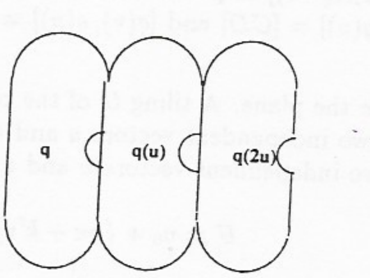

Fig. 2.5

Lemma 2.2.— *Let q, $q(u)$, $q(v)$ three instances of q, pairwise neighbouring. Then, q, $q(u)$, $q(v)$, $q(u-v)$, $q(-u)$, $q(-v)$, $q(v-u)$ don't overlap each other.* (Fig 2.6)

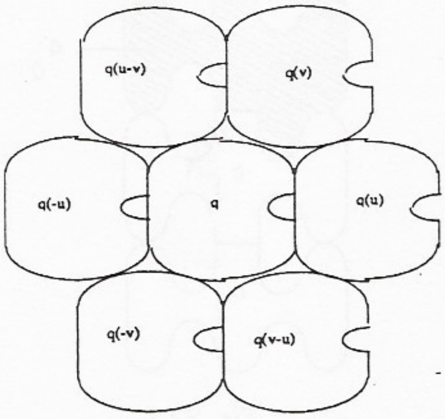

Fig. 2.6

Lemma 2.3.— *Let q be a tile q with boundary $b(q)$. Suppose there are two or three neighbouring instances of q that form a hole h (homeomorphic to a closed disk) (Fig.2.7). Then the size of the interior boundary I of these instances with respect to h is strictly less than $|b(q)|$.*

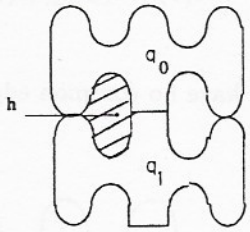

Fig. 2.7

Proof.— We give only the proof for two neighbouring instances of q because the modification of the proof of [10] is the same for two or for three tiles.

Suppose that $q(u)$ and $q(v)$ are neighbouring, and form a hole h. Let S be the set of edges belonging to the boundary of the hole h or both to $q(u)$ and $q(v)$. These edges have positive or null length. Let $l(I)$ be the length of the boundary I of h and $l(S)$ the sum of the lengths of the edges in S. Of course, $l(I) < l(S)$. We want to prove that two edges of S correspond to not overlapping arcs on the boundary of p. By this we mean that if $[A(u)B(u)]$ and $[C(v)D(v)]$ are two edges of S (possibly the same edge) then $[AB] \cap [CD]$ is empty or reduced to a point (in other way these arcs are not overlapping).

First of all, if $[A(u)B(u)]$ and $[C(v)D(v)]$ are the same edge of S then $[AB]$ and $[CD]$ are disjoined on the boundary of q: actually, $[DC] = [AB](v-u)$, and if $[AB]$ and $[DC]$ overlap along an arc $[MN]$ then the interior of q would be both on the left and on the right of $[MN]$; if $[AB]$ and $[DC]$ have a common extremity then we would have $B = C = B(v-u)$ or $A = D = A(v-u)$. So, $[AB]$ and $[DC]$ are disjoined. So we can suppose that $[A(u)B(u)]$ and $[C(v)D(v)]$ are distinct edges of S. Let $q_0 = q(u)$ $q_1 = q(v)$ $q_2 = q(2v-u)$. Because of lemma 2.1 q_0, q_1, q_2 do not overlap each other. The relative position of q_0 and q_1 is the same that q_1 and q_2. So we can extend both q_0 and q_1 to q_0' and q_1' in such a way that q_0' covers exactly q_0 and the hole h and q_1' covers q_1 and the corresponding hole (see figure 2.7.1).

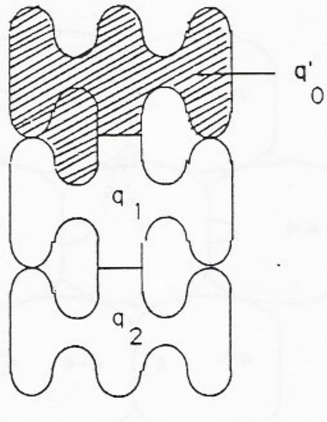

Fig. 2.7.1

Then again q'_0, q'_1, q_2 do not overlap each other. Suppose that $[A(u)B(u)]$, $[C(v)D(v)]$ are elements of S such that $[AB]$ and $[CD]$ are overlapping. Let $[MN]$ an arc of strictly positive length included in $[AB] \cap [CD]$. If $[A(u)B(u)]$ and $[C(v)D(v)]$ are both parts of q_0 and q_1 then q'_0 and q_2 border q'_1 along the same arc $[M(v)N(v)]$, which is a contradiction with the fact that q'_0 and q_2 do not overlap. The two other possibilities lead to the same kind of arguments.

Now if q_0 and q_1 have a common edge of strictly positive length the proof is achieved because we have:

$$l(I) < l(S) \leq |b(q)|$$

It remains the case where q_0 and q_1 have no common edge of strictly positive length.

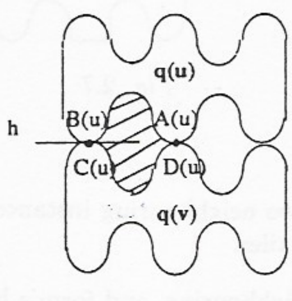

Fig. 2.7.2

In that case $l(I) = l(S)$. But in that case S contains only two edges of strictly positive length (figure 2.7.2) : $[A(u)B(u)]$ and $[C(v)D(v)]$ where $A(u) = D(v)$ and $B(u) = C(v)$.But $[AB]$ and $[DC]$ are disjoined (for the same reason that above) and $l(I) < |b(q)|$. ■

Lemma 2.4.— *Given a tile q with boundary $b(q)$. Let E be the exterior boundary of the union of any nonempty collection of instances of q which do not overlap each other and form no hole. Then $|E| \geq |b(q)|$.*

Proof.— Here again we have to refine the proof of [10]. Let $C = (q(u_1), ..., q(u_k))$ a not empty finite collection of instances of q which do not overlap each other and the union has no hole. We subdivise the boundary of q in a finite number of arcs in the following way: let c be the convex hull of q. There exists a finite sequence $\alpha_1, \alpha_2, ..., \alpha_n$ of points of $b(q) \cap b(c)$ such that the neighbouring of α_i in $b(q)$ is different of the neighbouring of α_i in $b(c)$ (Figure 2.8) (this is a consequence of the assumption we did concerning the tiles we deal with).

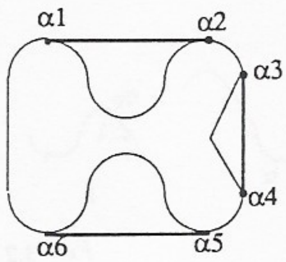

Fig. 2.8

If $[\alpha_i\alpha_{i+1}]$ is an arc of $b(q) \cap b(c)$ we insert between α_i and α_{i+1} a large enough number of points $\alpha_{i_1} = \alpha_i, ..., \alpha_{i_{n_i}} = \alpha_{i+1}$ in such a way that if β and γ are two consecutive points in this sequence, and if there exists an integer j such that $[\beta(u_j)\gamma(u_j)]$ overlaps an edge between $q(u_j)$ and another instance of q, $q(u'_j)$, then $[\beta(u_j)\gamma(u_j)]$ is entirely contained in this edge. This is possible because the number of edges is finite. We still call the new sequence $(\alpha_1, ..., \alpha_n)$.

Now we want to prove (and it is sufficient to prove that $l(E) \geq l(b(q))$ that for every i there exists j such that $[\alpha_i\alpha_{i+1}](u_j)$ is included in E. Let $s = [\alpha_i\alpha_{i+1}]$. We define a straight line l in the following way:

Case 1 If s is an arc of c then l is a perpendicular line to the line $\delta = (\alpha_i\alpha_{i+1})$

Case 2 If s is not an arc of c then l is a perpendicular line to the line δ cutting s in a single point m (s is a convex curve so it is possible).

The lines δ and l are oriented in such a way that q is on the left of δ and teh measure of the angle (l, δ) is $+\pi/2$. Now we project all the arcs $s(u_1), ..., s(u_n)$ on l (orientated as on the figure). Let p the rightmost point of these projections; p belongs to the projection of an arc $s(u_{k_0})$. Suppose that $s(u_{k_0}) \notin E$. There exists $q(u_k)$ such that $b(q(u_k) \cap s(u_{k_0}) \neq \emptyset$.

Case 1 Then $c(u_k)$ whose area is strictly larger than the area of $c(u_{k_0}) - q(u_{k_0})$ has to stick out of $c(u_{k_0}) - q(u_{k_0})$ and $s(u_k)$ has a projection beyond p, it contradicts the fact that p is the rightmost point on l.

Case 2 Then $b(q(u_k)) \supset s(u_{k_0})$, and again $c(u_k)$ lies beyond the line $\delta(u_{k_0})$ and it is also a contradiction.

■

3 Combinatorics and curves

We consider the set $\mathbf{C_0}$ of oriented curves which are a finite union of geometric arcs of class C^2 [1](some parts of the curve can be "multiple" arcs). We can define in $\mathbf{C_0}$ a product (partially defined) which is a natural operation: if a and b are curves such that the extremity of a is the origin of b then ab is the curve obtained as the "union" of a and b. Its origin is the origin of a and its extremity is the one of b. Clearly ab belongs to $\mathbf{C_0}$.

If we define in $\mathbf{C_0}$ the relation $a \sim b$ if there exists a translation t such that $t(a) = b$, then, clearly the above defined product is compatible with this equivalence relation and induces a product in the quotient set $\mathbf{C} = \mathbf{C_0}/\sim$ which has in this way a monoid structure. If α and β are two classes, the product $\alpha\beta$ is the equivalence class of the product of an element a of α with an element b of β such that the extremity of a is equal to the origin of b (Fig.3.1).

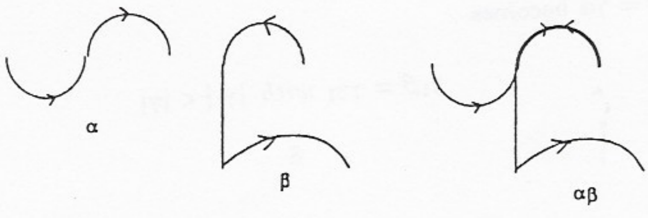

Fig. 3.1

This monoid C has combinatorial properties which look like the combinatorial properties of words of some alphabet; we will now establish them.

Elements of C will be called *curves* (to simplify the writing). The empty curve is denoted by 1.

A curve $\alpha \neq 1$ is said to be *primitive* if there exists no curve β such that $\alpha \in \beta\beta^+$.

Two curves α and β are *conjugate* if there exist two curves γ and δ such that $\alpha = \gamma\delta$ and $\beta = \delta$- (Fig.3.2).

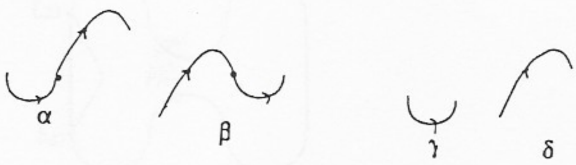

Fig. 3.2

Let α be a curve. The *mirror image* of α is the curve obtained by changing the orientation of α. The origin of α is the extremity of $\tilde{\alpha}$ and conversely.

Let $\alpha = \beta\gamma\delta$. Then γ is a *factor* of α, it is a *left factor* if $\beta = 1$ and a *right factor* if $\delta = 1$.

Lemma 3.1.— *If α is a factor of β^+ for a curve β of arbitrary small length, then α is a line segment.*

Proof.— It is a clear consequence of the fact that a real continuous and periodic function of real argument which has an arbitrary small period is a constant one. ■

Lemma 3.2.— *If $\alpha\beta = \gamma\alpha$ there exists a curve $\delta = \delta_1\delta_2$ and a conjugate $\delta_c = \delta_2\delta_1$ such that $\alpha \in \delta^*\delta_1$, $\beta \in \delta_c^+$ and $\gamma \in \delta^*$.*

Proof.—

- If $|\alpha| \leq |\gamma|$, we have:

$$\gamma = \alpha\lambda$$

and

$$\beta = \lambda\alpha$$

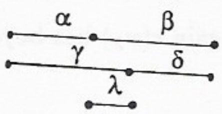

Fig.

So the result is proved, if we write: $\delta = \alpha\lambda$ and $\delta_c = \lambda\alpha$

- If $|\alpha| \geq |\gamma|$ we have:

$$\alpha = \gamma^k\gamma_1$$

with γ_1 a strict left factor of γ.

Fig.

So the equality $\alpha\beta = \gamma\alpha$ becomes

$$\gamma_1\beta = \gamma\gamma_1 \textit{ with } |\gamma_1| < |\gamma|$$

Then β and γ are conjugate and we have

$$\gamma = \gamma_1\gamma_2 \ , \ \beta = \gamma_2\gamma_1 \ \ \textit{and} \ \alpha \in \gamma^*\gamma_1.$$

■

Lemma 3.3.— *If α and β are nonempty curves satisfying the equation*

$$\alpha^p = \beta^q \ p, q > 0$$

then there exists a curve γ such that $\alpha, \beta \in \gamma^+$.

Proof.—

If $|\alpha| = |\beta|$ then $p = q$ and $\alpha = \beta = \gamma$.

Assume that $|\alpha| > |\beta|$.

Then $\alpha = \beta^k\beta_1$ with $|\beta_1| < |\beta|$.

If $\beta_1 = 1$, the proof is achieved.

Otherwise, β_1 is a left factor and a right factor of α. In the same way, we can write:

$\alpha = \beta_2\beta^k$ with $|\beta_2| < |\beta|$.

And β_2 is also a left and a right factor of α.

But $|\beta_2| = |\beta_1|$. So $\beta_2 = \beta_1$, and the following equality holds:

$$\beta_1\beta^k = \beta^k\beta_1$$

By lemma 3.2, β_1 and β^k are power of a same curve γ.

And $\alpha = \beta_1\beta^k \in \gamma^+$.

Now we have an equation

$\gamma^{p_1} = \beta^q$ *and* $|\gamma| < |\beta|$, $p_1 > p$.

We iterate the process. If it does not stop, that means that α^p is a power of infinite number of curves, so by lemma 3.1, α^p is a line segment, and the result is proved in all the cases. ■

Lemma 3.4.— *If $\alpha\beta = \beta\alpha$ then α and β are both powers of some curve γ, or $\alpha\beta$ is a line segment.*

Proof.—

If $\alpha = 1$ or $\beta = 1$ it is clear (if we put $\alpha^0 = 1$).

Let us suppose α and $\beta \neq 1$.

If $|\alpha| = |\beta|$ then $\alpha = \beta = \gamma$. Let us suppose $|\alpha| > |\beta|$. Then $\alpha = \beta^k\beta_1 = \beta_1\beta^k$ and $0 \leq |\beta_2| < |\beta_1|$. If $|\beta_1| = 0$, the result is proved. In the other case, we iterate the process for β_1 and β. We have $\beta_1\beta = \beta\beta_1$. So, $\beta = \beta_1^{k'}\beta_2$ with $0 \leq |\beta_2| < |\beta_1|$. We can observe that $|\beta_2| \leq 1/2\,|\beta|$. If the process does not stop, the sequence $|\beta_1|$ is converging to zero, so $\alpha\beta$ satisfies the hypothesis of lemma 3.1 and we get the result. ■

Lemma 3.5.— *If α and β are primitive curves such that α^+ and β^+ have a common element γ of length greater than or equal to $|\alpha| + |\beta|$, then $\alpha = \beta$.*

Proof.—

Let γ a curve such that $\gamma = \alpha^p = \beta^q$, with $|\gamma| \geq |\alpha| + |\beta|$. Suppose that $|\alpha| > |\beta|$,then $1 < p < q$. So we have $\alpha = \beta^k\beta_1$ with $0 < |\beta_1| < |\beta|$ because α is a primitive curve. Let us write $\beta = \beta_1\beta_2$. But, β_1 is a left and right factor of β and also β_2 because α^2 is a left factor of γ. So, $\beta_1\beta_2 = \beta_2\beta_1$ and there is a contradiction with lemma 3.4. ■

Lemma 3.6.— *Let α be a primitive curve. If a conjugate β of α has two different writings $\beta = \beta_1\beta_2 = \beta'_1\beta'_2$ such that $\alpha = \beta_2\beta_1 = \beta'_2\beta'_1$ then*

$$\beta_1 = 1 \text{ and } \beta'_1 = \beta \text{ or viceversa}$$

Proof.—

The equality $\beta_2\beta_1 = \beta'_2\beta'_1$ implies $\beta_1\beta_2\beta_1\beta'_2 = \beta_1\beta'_2\beta'_1\beta'_2$.

Whence $(\beta'_1\beta'_2)(\beta_1\beta'_2) = (\beta_1\beta'_2)(\beta'_1\beta'_2)$.

By lemma 3.4, there exists γ such that $\beta_1\beta'_2$ and $\beta'_1\beta'_2$ belong to γ^+. In the same way:

$\beta'_1\beta_2\beta_1\beta_2 = \beta'_1\beta'_2\beta'_1\beta_2 = \beta'_1\beta'_2\beta'_1\beta_2$ so $(\beta'_1\beta_2)(\beta_1\beta_2) = (\beta'_1\beta_2)(\beta'_1\beta_2)$.

So we have:$\beta_1\beta_2 = \beta'_1\beta'_2 = \gamma$.

And $\beta'_1\beta_2,\ \beta_1\beta'_2 \in \gamma^*$. But one of the two curves $\beta'_1\beta_2$ or $\beta_1\beta'_2$ has a length strictly less than γ because $\beta_1 \neq \beta'_1$ so $\beta'_1\beta_2 = 1$ or $\beta_1\beta'_2 = 1$. And the result is proved. ■

4. Tilings

Our purpose is to characterize the exact tiles by a simple property, and to describe all the tilings we can obtain with an exact tile.

First of all we can observe that the following properties hold:

Lemma 4.1.—

Let q be an exact tile and U a tiling by q. Then two instances $q(u), q(v)$ $(u, v \in U)$ are disjoint or simply neighbouring.

Proof.—

If two instances are neighbouring but not simply neighbouring their union form a hole and by lemma 2.3 and 2.4 they cannot belong to a tiling of the plane. ■

Lemma 4.2.—

Let U be a tiling of the plane by an exact tile q. If $q(u)$ is an instance of q in the tiling, every surrounding of $q(u)$ with tiles of U has at least four tiles.

Proof.— It is an immediate consequence of lemma 2.3 and 2.4. ■

Triads and contacts.—

A *triad* is a triple $(q(u), q(v), q(w))$ of tiles which are two by two simply neighbouring and $[q(u), q(v)]$, $[q(u), q(w)]$ are consecutive curves on the boundary of $q(u)$ in this order, and moreover, the union of the three tiles has no hole. This implies the existence of a unique common point to the three tiles.

If we have :

$$[q(u), q(v)] = [A_1A],\ \ [q(v), q(w)] = [B_1B],\ \ [q(w)q(u)] = [C_1C]$$

then (A, B, C) will be called the *contact* of the triad $(q(u), q(v), q(w))$.

(Fig.4.1 $q(u) = p,\ q(v) = q,\ q(w) = r$)

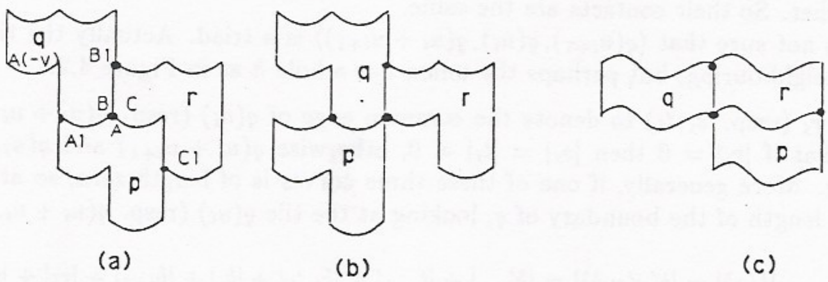

Fig. 4.1

We can observe that if (p,q,r) is a triad with contact (A,B,C), then (q,r,p) and (r,p,q) are also triads with respective contacts (B,C,A) and (C,A,B).

A contact is said to be *exact* if there is at least one tiling of the plane by the corresponding tile, in which this contact appears. In Figure 4.1, (a) and (c) but not (b) are exact contacts.

We will give now a characterization of exact tiles.

Theorem 4.2.— *A tile q is exact if and only if it admits a surrounding.*

Proof.— Clearly, if U is a tiling of the plane by a tile q, then if $q(u)$ is a tile of U, the set of tiles of U which intersect $q(u)$ are simply neighbouring to $q(u)$ (Lemma 4.1) and form a surrounding of $q(u)$ when correctly ordered according to their common edge with $q(u)$.

Conversely, let us suppose that $(q(u_0),...,q(u_{k-1}))$ is a surrounding of q. Because of lemma 4.2, k is greater than or equal to four. Let us write:

$e_i = [q, q(u_i)]$ and $e'_i = [q(u_i), q(u_{i+1})]$ (indices are defined modulo k).

Using translations one has:

$$[q, q(u_i)] = [q(u_i), q(2u_i)] = [q(u_{i+1}, q(u_i + u_{i+1})] = e_i$$

$$[q, q(u_{i+1})] = [q(u_i), q(u_i + u_{i+1})] = [q(u_{i+1}), q(2u_{i+1})] = e_{i+1}$$

$$[q(u_i), q(u_{i+1})] = [q(2u_i), q(u_i + u_{i+1})] = [q(u_i + u_{i+1}), q(2u_{i+1})] = e'_i$$

We have represented in Figure 4.2 the tiles q, $q(u_i)$, $q(u_{i+1})$, $q(u_{i-1})$, $q(u_{i-1} + u_i)$, $q(2u_i)$, $q(u_i + u_{i+1})$, $q(2u_{i+1})$ and their common edges.

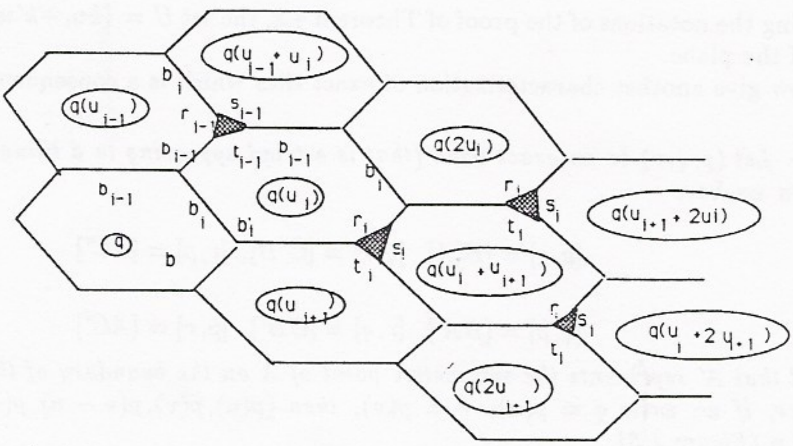

Fig. 4.2

The triads $(q, q(u_i), q(u_{i+1}), (q(u_i), q(2u_i), q(u_i + u_{i+1}), (q(u_{i+1}), q(u_i + u_{i+1}), q(2u_{i+1}))$ are translated one of the other. So their contacts are the same.

But it is not sure that $(q(u_{i+1}), q(u_i), q(u_i + u_{i+1}))$ is a triad. Actually the tiles, of course, are two by two simply neighbouring, but perhaps the union has a hole h as in Figure 4.2.

Let us write r_i (resp. s_i, t_i) to denote the common edge of $q(u_i)$ (resp. $q(u_i + u_{i+1}), q(u_{i+1})$) with h. Let us observe that if $|r_i| = 0$ then $|s_i| = |t_i| = 0$, otherwise $q(u_i + u_{i+1})$ and $q(u_{i+1})$ would not be simply neighbouring. More generally, if one of these three curves is of length zero, so are the others. Now let us compute the length of the boundary of q, looking at the tile $q(u_i)$ (resp. $q(u_i + u_{i+1})$). We have:

$$|b(q)| = |b(q(u_i))| = |b'_{i-1}| + |t_{i-1}| + |b_{i-1}| + |b_i| + |b_{i+1}| + |r_i| + |b'_i| + |b_i|$$

If we sum these equalities for $i = 0$ to $k-1$, we obtain:

$$k\,|b(q)| = 2\sum |b'_i| + 4\sum |b_i| + \sum |t_i| + \sum |r_i| \quad (1)$$

On the other hand, computing the length of $b(q)$ looking at $q(u_i + u_{i+1})$ (Figure 4.2), we obtain:

$$|b(q)| = |b_i| + |s_i| + |b_{i+1}| + |b'_i| + |t_i| + |b_i| + |b_{i+1}| + |r_i| + |b'_i|$$

And also

$$k\,|b(q)| = 4\sum |b_i| + 2\sum |b'_i| + \sum |s_i| + \sum |t_i| + \sum |r_i| \quad (2)$$

Comparing **(1)** and **(2)**, one has

$$\sum |s_i| = 0$$

So for each i, $|s_i| = |t_i| = |r_i| = 0$, and the hole h is reduced to a point. So $(q(u_{i+1}), q(u_i), q(u_i + u_{i+1}))$ is a triad for each i. Moreover

$$(q(u_i - u_{i+1}), q(u_i), q(u_{i+1}), q(u_{i+1} - u_i), q(-u_i), q(-u_{i+1})) \quad (3)$$

is a surrounding of q (because of lemma 2.2).

This surrounding has the property it can be translated so the set $U = \{nu_i + n'u_{i+1}/n, n' \in \mathbb{Z}\}$ is a **regular tiling** of the plane.

So the proof is achieved and it contains some more properties we state below. ∎

Corollary 4.3.— *If a tile q is exact, there exists a regular tiling of the plane by q.*

Corollary 4.4.— *Every surrounding of a tile can be extended into a tiling of the plane, and every contact appearing in a surrounding is an exact one.*

Proof.— Keeping the notations of the proof of Theorem 4.2, the set $U = \{ku_i + k'u_{i+1}/i = 0, ..., k-1,\ n, n' > 0\}$ is a tiling of the plane. ∎

We can now give another characterization of exact tiles which is a consequence of **(3)**.

Lemma 4.5.— *Let (p, q, r) be an exact triad (that is a triad appearing in a tiling of the plane) with contact (A, B, C). Then we have*

i)

$$[p, q] = [B'A],\ [q, r] = [C'B],\ [r, p] = [AC']$$

$$[q, p] = [BA'],\ [r, q] = [CB'],\ [p, r] = [AC']$$

(We recall that A' represents the symmetric point of A on the boundary of the tile)

ii) Moreover, if we write $q = p(u)$, $r = p(v)$, then $(p(u), p(v), p(v-u), p(-u), p(-v), p(u-v))$ is a surrounding of p. (Figure 4.3)

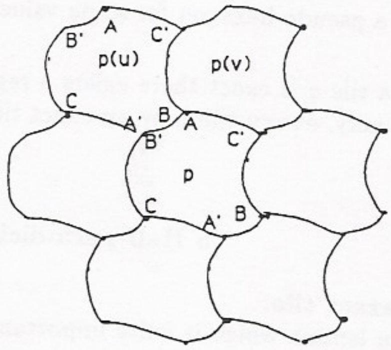

Fig. 4.3

Corollary 4.6.— *If $[p,q] = [AB]$ is an edge in a tiling U then $[q,p] = [A'B']$.*

Let q be a tile , and A, B two points of its boundary. From now, the equivalence class in $C = C_0/\sim$ of the curve $[AB]$ will be denoted by $< [AB] >$.

Definition 4.7.— A tile q is a *pseudo-hexagon* $ABAA'B'C'$ if there exist three points A, B and C on the boundary of q such that $B \in [AC]$ and $< [A'B'] > = < [AB] > < [B'C'] > = < [BC] > < [C'A] > = < [CA'] >$.

This is equivalent to saying that there exists a point A of the boundary of q such that the equivalence class of the boundary of q considered as a Jordan curve starting and ending at A can be written as $\alpha\beta\gamma\tilde{\alpha}\tilde{\beta}\tilde{\gamma}$ (Figure 4.4).

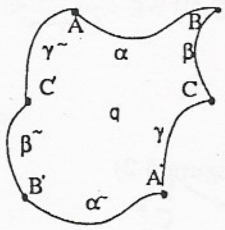

Fig. 4.4

The previous result can be stated in a new way.

Theorem 4.8.— *A tile q is exact if and only if it is a pseudo-hexagon.*

Proof.— The "only if" part is proved by using Lemma 4.5. Conversely, suppose q is a pseudo-hexagon $ABCA'B'C'$. Let $u = \overrightarrow{AC}, v = \overrightarrow{BA'}$. Then $(q(u), q(v), q(v-u), q(-u), q(-v), q(u-v))$ is a surrounding of q and so q is an exact tile. ∎

As an immediate consequence of Lemma 4.5, one has

Proposition 4.9.— *If (A, B, C) is an exact contact for the tile q then q is a pseudo-hexagon $(B'AC'BA'C)$ and (A', B', C') is also an exact contact.*

In Theorem 4.8 we give an "if and only if" condition for a tile to be an exact one. The question arising now is the effectiveness of this condition. Actually this condition is decidable for a large class of tiles. Let us consider a tile as defined by two real functions f and g such that $x = f(s)$ and $y = g(s)$ are the parametric equations of the boundary (where s is the curvilinear abscissa of the boundary);let us denote the tile by $q(f,g)$. Then we have :

Theorem 4.10.— *Let P be the class of real functions of a real variable which are piecewise polynomial functions. Let $q(f,g)$ be a tile such that f and g belong to P. Then the exactness of q is decidable.*

Proof.— Let l the length of q. Then f and g are defined on the segment $[0,l]$. There exists a sequence $(s_1 = 0, .., s_n = l)$ such that f and g are polynomial functions in each segment $[s_i, s_{i+1}]$. So, there exists a finite number of choices for three numbers a, b, c in $[0, l/2]$ if we consider only the choice of the segments

$[s_i, s_{i+1}$ in which the numbers are located. Given this choice, one can decide whether $M(a)M(b)M(c)M(a+l/2)M(b+l/2)M(c+l/2)$ is a pseudo-hexagon for some value of a, b, c.

We have proved that if a tile q is exact there exists a regular tiling of the plane by q. But, a stronger property can be proved. Actually, every tiling by an exact tile is half-periodic. That is the aim of the next part.

5 Half-periodicity

In this part, q is an exact tile.
We first establish a main lemma which is quite important for the proof of half-periodicity.

Lemma 5.1.— *If (A, C, B') and (A, D, B') are two different exact contacts ($C \neq D$) such that $C \in [BD]$ and ($B \neq C$ or $A' \neq D$) then one of the following properties is satisfied:*
i) there exists a primitive curve $\alpha = \alpha_1\alpha_2 \in C$ and a conjugate $\alpha_c = \alpha_2\alpha_1$ of α such that

$$< [BA'] > \in \alpha\alpha^+, \quad < [B'A] > \in \widetilde{\alpha}^+$$

$$< [CD] > \in \alpha_c^+, \quad < [C'D'] > \in \widetilde{\alpha^+}$$

$$< [BC] > \in \alpha^*\alpha_1, \quad < [B'C'] > \in \widetilde{\alpha_1}\widetilde{\alpha^*} = \widetilde{\alpha_c^*}\widetilde{\alpha_1}$$

$$< [DA'] > \in \alpha_2\alpha^*, \quad < [D'A] > \in \widetilde{\alpha^*}\widetilde{\alpha_2} = \widetilde{\alpha_2}\widetilde{\alpha_c^*}$$

(Figure 5.1)

or

ii)$[BA']$ is a line segment. (Figure 5.2)

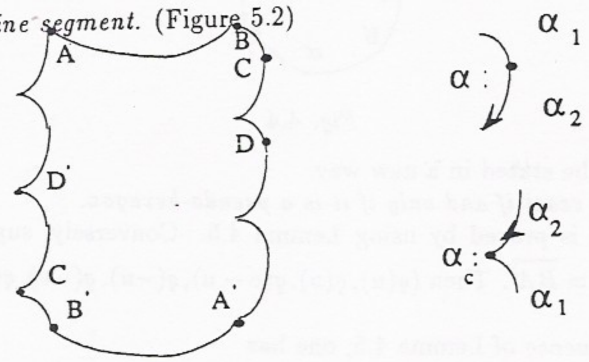

Fig. 5.1

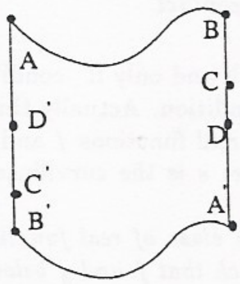

Fig. 5.2

Proof.— Let $<[BC]>=a \quad <[CD]>=b \quad <[DA']>=c$.

Then $<[CA']>=<[CD]><[DA']>=bc$.

One has $<[C'A]>=<\widetilde{[CA']}>=\widetilde{(bc)}=\tilde{c}\tilde{b}$.

On the other hand

$$<[C'A]>=<[C'D'][D'A]>=<[C'D]><\widetilde{[DA']}>=<[C'D']>\tilde{c}$$

Let $<[C'D']>=d$ $(d \neq 1)$. We get the equation

$$\tilde{c}\tilde{b}=d\tilde{c}$$

By lemma 3.2, there exists $\alpha=\alpha_1\alpha_2 \in C$ and a conjugate $\alpha_c=\alpha_2\alpha_1$ of α such that

$$\tilde{c} \in \alpha^*\alpha_1, \quad \tilde{b} \in \alpha_c^+, \quad d \in \alpha^+. \quad (4)$$

On the other hand

$$<[BD]>=<[BC]><[CD]>=ab \text{ and } <[B'D']>=<\widetilde{[BD]}>=\tilde{b}\tilde{a}$$

but also $<[B'D']>=<[B'C']><[C'D']>=\tilde{a}<[C'D']>$.

So $\tilde{b}\tilde{a}=\tilde{a}d$.

Hence there exists a curve $\alpha'=\alpha'_1\alpha'_2 \in C$ and a conjugate $\alpha'_c=\alpha'_1\alpha'_2$ such that

$$\tilde{b} \in \alpha'^{+}_c \quad \tilde{a} \in \alpha^*_c\alpha'_2=\alpha'_2\alpha'^* \text{ and } d \in \alpha'^+ \quad (5)$$

If α is a line segment, so is α' and ii) holds.

If α and α' are not line segments, we can suppose in (4) and (5) that α is a primitive curve and also α'. So $\alpha=\alpha'$ by lemma 3.6. And consequently $\alpha_c=\alpha'_c$. So by lemma 3.7 it implies that $\alpha_1=\alpha'_1$ and $\alpha_2=\alpha'_2$, or $\alpha_1=\alpha'_2=1$ or $\alpha_2=\alpha'_1=1$.

In the three cases, one has:

$$<[BA']>=<[BC]><[CA']>\in \widetilde{\alpha'_2}\widetilde{\alpha^*_c\alpha^+_c}\widetilde{\alpha_1}\widetilde{\alpha^*}=\tilde{\alpha}\widetilde{\alpha^+}$$

In the same way we have: $<[B'A]>\in \alpha_c\alpha^+$.

So the result i) is proved (changing α in α_c) and can be explicited by the following scheme obtained by an "unfolding" of the boundary of q (Figure 5.3).

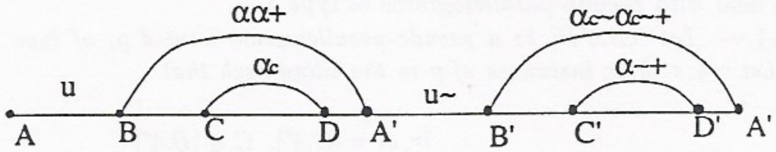

Fig. 5.3

We complete this result by the following obvious remark:

Lemma 5.2.— *If (A,B,B') is an exact contact, then we have $<[AB]>=<\widetilde{[A'B']}>$ and $<[BA']>=<\widetilde{[B'A]}>$ (Figure 5.4) and (A,A',B') is also an exact contact.*

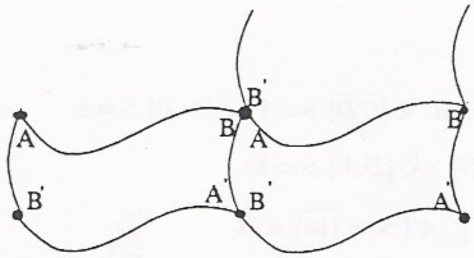

Fig. 5.4

Definition 5.3.— A tile q is a *pseudo-parallelogram* $ABA'B'$ if there are two points A, B on the boudary of q such that $B \in [AA']$ and either

i) $< [AB] >=< \widetilde{[A'B']} >$ and there is a curve α and a conjugate α_c such that $< [BA'] > \in \alpha\alpha^+$ and $< [B'A] > \in \widetilde{\alpha_c \alpha_c^+}$ (the tile will be called a pseudo-parallelogram of type 1 or :

ii) $< [AB] >=< \widetilde{[A'B']} >$ and $< [BA'] >=< \widetilde{[B'A]} >$ (the tile will be called a pseudo-parallelogram of type 2) (Figure 5.5)

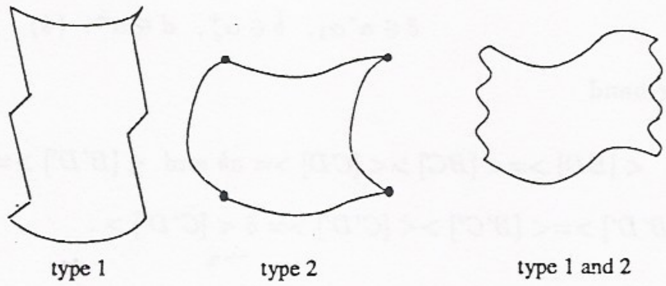

Fig. 5.5

One can observe that a pseudo-parallelogram is a special kind of exact tile, and that a pseudo-parallelogram may have types 1 and 2.

Lemma 5.3.1.— *If a tile q is a pseudo-parallelogram $ABA'B'$ and also a pseudo-parallelogram $ACA'C'$ then $B = C$.*

Proof.— Let us write: $< [AB] >= u$, $< [BC] >= v$ and $< [CA'] >= w$. Suppose that $|[AB]| < |[AC]|$. There exist two curves w_1 and v_1 such that $vw = w_1 v_1$ and $|v| = |v_1|$.

Since q is a pseudo-parallelogram $ABA'B'$ and also $ACA'C'$, the following equalities hold:

$< \widetilde{[AC]} > =< [A'C'] >$. So, $\tilde{v}\tilde{u} = \tilde{u}\tilde{v_1}$. And also, $< \widetilde{[BA']} > =< [B'A] >$. So, $\tilde{w}\tilde{v} = \tilde{v_1}\tilde{w}$.

Using lemma 3.2 and lemma 3.6, we obtain:

there exists a curve $\gamma = \gamma_1\gamma_2$ such that $u \in \gamma^*\gamma_1$, $v \in (\gamma_2\gamma_1)^+$, $v_1 \in \gamma^+$ and $w \in (\gamma_2\gamma_1)^*\gamma_2$. Now we have $uvw \in \gamma\gamma^+$. So $uvw\tilde{u}\tilde{v}\tilde{w}$ admits a strict factor which is a closed curve, so there is a contradiction. ∎

Lemma 5.1 and 5.2 have a corollary.

Corollary 5.3.2.— *If (A, C, B') and (A, D, B') are two different contacts then q is a pseudo-parallelogram and conversely.*

We give now four technical lemmas about pseudo-parallelograms which will be needed later. The first three lemma deal with pseudo-parallelograms of type 1.

Lemma 5.4.1.— *Let $ABA'B'$ be a pseudo-parallelogram named p, of type 1, and let U be a tiling of the plane by p. Let r, q, s three instances of p in the tiling such that*

$$[r, q] = [CA'], \ C \in [BA']$$

$$[q, s] = [DA'], \ D \in [BA'], \ D \neq B$$

$$C \neq D \textit{ and } |[B'C']| \geq |\alpha|$$

where α *is a curve such that* $< [BA'] > \in \alpha\alpha^+$ *(Figure 5.6).*

Under these hypothesis, the tile $r(\overrightarrow{AB'})$ *belongs to the tiling*

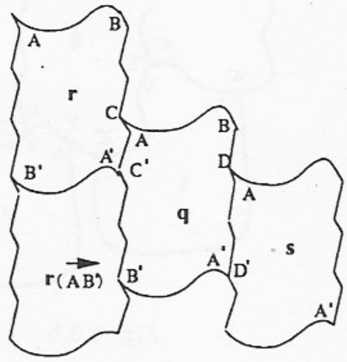

Fig. 5.6

Proof.— Let t be the tile adjacent to r in the surrounding of q (before r) in the tiling U. If $t \neq r(\overrightarrow{AB'})$ then the contact of the triad (t, r, q) is (X, A', C') with $X \neq B$. There are two cases to examine according to $X \in [AB]$ or $X \in [BC]$.

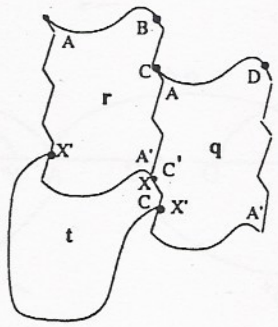

Fig. 5.6.a

<u>*First case :*</u> (Figure 5.6.a) $X \in [BC]$

The tile p admits two different contacts (C', X, A') and (C', B, A'), and $B \in [AX]$. So, we apply lemma 5.1 ($B \neq A$). If ii) holds then $< [AC] >$ is a line segment with factor α, so $< [AA'] >$ is a line segment, it is impossible. So, i) holds. There exists a primitive curve β and a conjugate β_c such that $< [AC] > \in \beta\beta^+$ and $< [BX] > \in \beta_c^+$. So we obtain the scheme below:

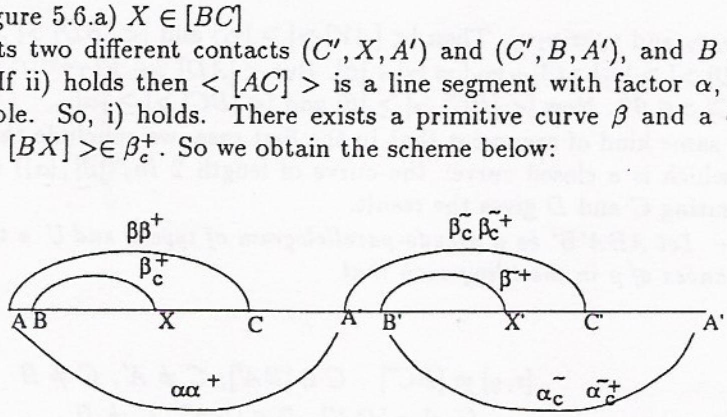

Fig. 5.6.a1

But let us recall that $|[BC]| \geq |\alpha|$. If $|\beta| \geq |\alpha|$ then β_c which is a left factor of $< [BX] >$ has all his factors of length $|\alpha|$ which are conjugate curves of α. So the curve of length $2|\alpha|$ centered in A' is a closed curve because its equivalence class is α, and it cannot be a factor of the boundary of p.

If $|\alpha| \geq |\beta|$ the argument is symetric.
<u>Second case</u> (Figure 5.6.b) $X \in [AB]$

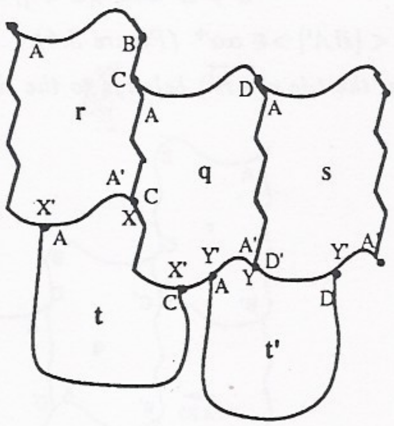

Fig. 5.6.b

This case is less easy than the previous one. Let us observe that $X \neq A$ otherwise r and t would be no longer adjacent. So there exists an instance t' adjacent to s in the surrounding of q (after s). Let (Y, A', D') be the contact of the triad (t', q, s). But Y is distinct of B so there are two distinct contacts (D', Y, A') and (D', B, A'), with $D \neq B$. If ii) of lemma 5.1 holds $< [AD] >$ is a line segment, so also α because $|[BC]| \geq |\alpha|$ and it is impossible because $[AA']$ is not a segment line. It implies that $< [AD] > \in \gamma\gamma^+$ (γ is primitive). So we have (assuming $D \in [BC]$):

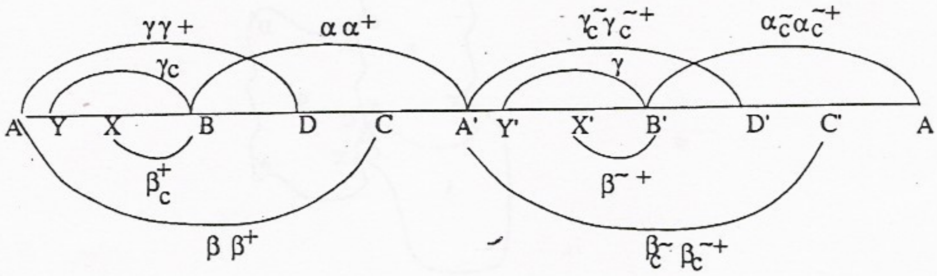

Fig. 5.6.b1

Let $\gamma = \gamma_1\gamma_2$ and $\gamma_c = \gamma_2\gamma_1$. Then $|< [AY] >| \geq |\gamma_1|$ and $|< [BD] >| \geq |\gamma_2|$.
So $|< [AD] >| \geq |\gamma_1| + |\beta| + |\gamma_2| = |\gamma| + |\beta|$. But $< [AD] > \in F(\gamma\gamma^+) \cap F(\beta\beta^+)$, so (lemma 3.5) $\beta = \gamma$. It implies $< [DC] > \in \beta^+$. Now, $|< [BC] >| \geq |\beta|$ and $|< [BC] >| \geq |\alpha|$.

With the same kind of argument that in the first case, we conclude there exists a strict factor of the boundary of p which is a closed curve: the curve of length 2 $inf(|\beta|, |\alpha|)$ centered in A'. If we suppose $C \in [BD]$, permuting C and D gives the result. ∎

Lemma 5.4.2.— *Let $ABA'B'$ be a pseudo-parallelogram of type 1 and U a tiling of the plane by p. Let r, q and s three instances of p in the tiling such that*

$$[r, q] = [BC'] \ , \ C \in [BA'], \ C \neq A', \ C \neq B$$
$$[q, s] = [DA'], \ D \in [BA'], \ C \neq D$$

Then the tile $q(\overrightarrow{AB'})$ belongs to the tiling U.(Figure 5.7.1)

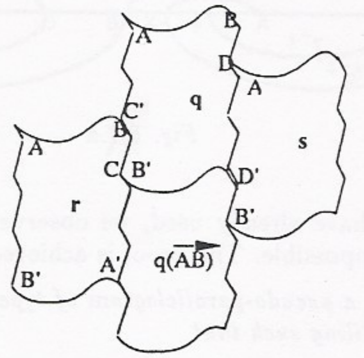

Fig. 5.7.1

Proof.— Let us assume that the tile adjacent to r in the surrounding of q (before r) is not $q(\overrightarrow{AB'})$, but a tile t such that the contact (t, r, q) is (X, C, B') with $X \neq A$ (Figure 5.7.2). Necessarily $X' \in [A'B']$, so $X \in [AB]$.

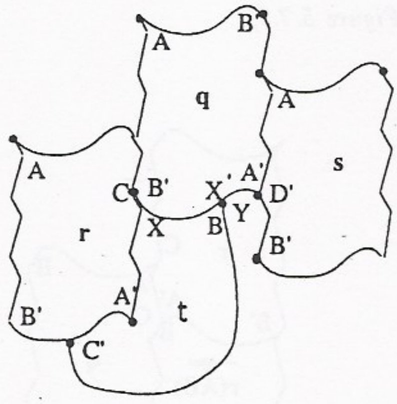

Fig. 5.7.2

So, there exists a tile t' adjacent to t (before t) in the surrounding of q with a contact (Y, B, X') for the triad (t', t, q). We cannot have $Y = C'$ because $Y' \in [A'X']$. So we have the following inequalities about exact contacts:

$$(B', A, C) \neq (B', X, C) \text{ and } C' \neq A$$
$$(X', Y, B) \neq (X', C', B) \text{ and } C \neq B$$

We apply lemma 5.1 at these two inequalities. In both cases, ii) cannot hold because it implies $[BB']$ is a line segment.

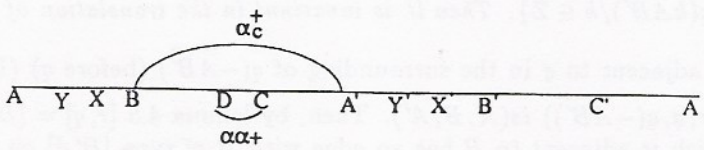

Fig. 5.7.2a

In the same way, we can prove that we cannot have property i) for an inequality and ii) for the other. So, property i) holds for the two inequalities, and we have the following scheme:

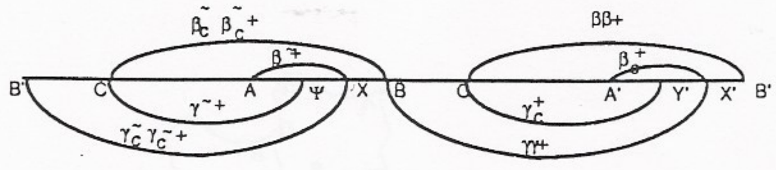

Fig. 5.7.a

Then by an argument we have already used, we observe that the curve centered in B with length $2inf(|\beta|, |\gamma|)$ is closed and it is impossible. The proof is achieved. ■

Lemma 5.4.3.— *Let $ABA'B'$ be a pseudo-parallelogram of type 1 and U a tiling of the plane by p. Let r, q and s three instances of p in the tiling such that*

$$[r,q] = [CA'] \ , \ C \in [BA']$$

and $|[BC]|$ is greater than or equal to the upperbound of the lengths of the edges of the tiling. Then $r(\overrightarrow{AB'})$ belongs to the tiling U. (Figure 5.7.3).

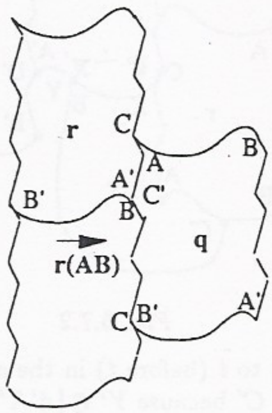

Fig. 5.7.3

Proof.— The proof is similar to the one of lemma 5.4.1. But the hypothesis about $|[BC]|$ implies that the tile t cannot satisfy the hypothesis of the second case ($X \in [AB]$) because in that case, the edge $[XC] = [t,q]$ would have a length larger than the upperbound, so t satisfies the hypothesis of case one ($X \in [BC]$) and it provides a contradiction (without using the tile s). ■

Lemma 5.4.4.— *Let $ABA'B'$ be a pseudo-parallelogram of type 1, and U a tiling such that U contains the biinfinite band $B = \{q(k\overrightarrow{AB'})/k \in \mathbb{Z}\}$. Then U is invariant in the translation of vector $\overrightarrow{AB'}$.*

Proof.— Let r a tile adjacent to q in the surrounding of $q(-\overrightarrow{AB'})$ (before q) (Figure 5.7.4) such that the contact of the triad $(r, q, q(-\overrightarrow{AB'}))$ $is(X, B, A')$. Then, by lemma 4.5 $[r,q] = [B'X]$, $[r, q(-\overrightarrow{AB'})] = [XA]$. So every tile of U which is adjacent to B has an edge with B of type $[B'A]$ on one side and $[AB']$ on the other side (symetrically). That proves that all the tiles $r(k\overrightarrow{AB'})$ belong to U. So U is a union of bands translated of B. And, so U is invariant by the translation of vector $\overrightarrow{AB'}$. ■

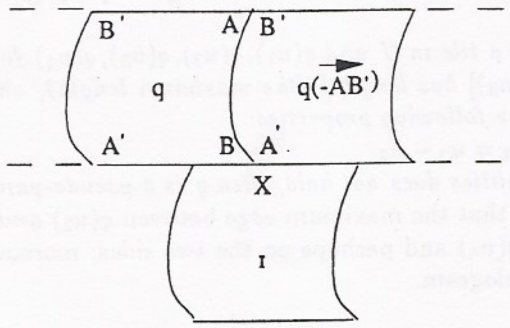

Fig. 5.7.4

Lemma 5.4.5.— *Let $ABA'B'$ be a pseudo-parallelogram p where $[BA']$ is a line segment. If a tiling U contains two adjacent instances of p q and r such that*

$$[q, r] = [CA'],\ C \in [BA'] \text{ and } C \neq B$$

(Figure 5.7.5a)

then $q(\overrightarrow{AB'}$ belongs to the tiling or there exists a primitive curve $\gamma = \gamma_1\gamma_2$ and a conjugate $\gamma_c = \gamma_2\gamma_1$ $(\gamma_1, \gamma_2 \neq 1)$ such that $< [AC] > \in \gamma\gamma^+$ and $< [BC] > = \gamma_2$.

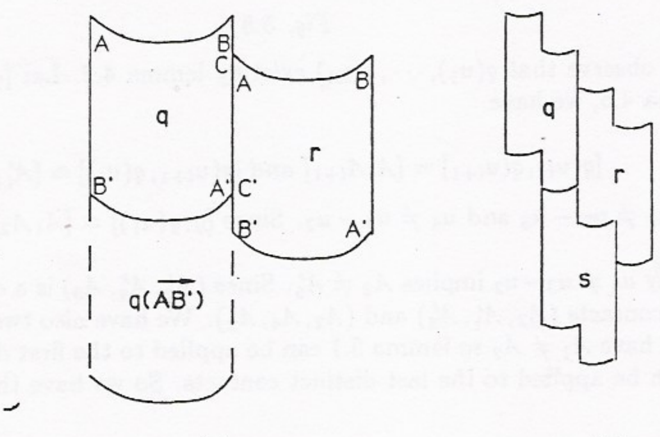

Fig. 5.7.5

Proof.— Let s be the tile adjacent to r and neighbouring of q (before q) in U. Let (C', X, A') the contact of the triad (r, s, q). If $X \neq B$ the lemma 5.1 can be applied, so if ii) holds then $[AC]$ is a line segment, and it is impossible. Whence i) holds: there is aprimitive curve $\gamma = \gamma_1\gamma_2$ and a conjugate $\gamma_c = \gamma_2\gamma_1$ of γ such that $< [AC] >$ belongs to $\gamma\gamma^+$ and $< [XB] >$ (or $< [BX] >$ it depends on the place of X) belongs to γ_c^+.

If $X \in [BC]$ then $[AC]$ is a line segment,it is impossible.

If $X \in [AB]$ then we have necessarily

$$< [AX] > \in \gamma^*\gamma_1 \quad < [XB] > \in (\gamma_2\gamma_1)^+ \text{ and } < [BC] > = \gamma_2.$$

So the result is proved. ■

Theorem 5.5.— *If p is an exact tile, every tiling of the plane by p is half-periodic.*

Proof.— Let U be a tiling of the plane by p. Let l be the upperbound of the lengths of the edges in U. There are two possibilities. There exists an edge of length l or all the edges have a length strictly less than l.

We first treat the former case.

A) There exists in the tiling an edge of length l.

The first step is to prove that the maximum edge is "propagated" in the tiling, and this in two possible ways.

Lemma 5.6.— *Let q be a tile in U and $q(u_1), q(u_2), q(u_3), q(u_4)$ for consecutive tiles in a surrounding of q in U such that $[q(u_2), q(u_3)]$ has length l (the maximum length), and $q(u_1), q(u_4)$ are adjacent to q (Figure 5.8). Then U satisfies the following properties:*

i) $u_1 = u_2 - u_3$ or $u_4 = u_3 - u_2$

ii) If one of both equalities does not hold, then q is a pseudo-parallelogram.

The part i) proves that the maximum edge between $q(u_2)$ and$q(u_3)$ is propagated at least on one side between q and $q(u_1)$ or $q(u_4)$ and perhaps on the two sides; moreover if it is propagated only on one side, then q is a pseudo-parallelogram.

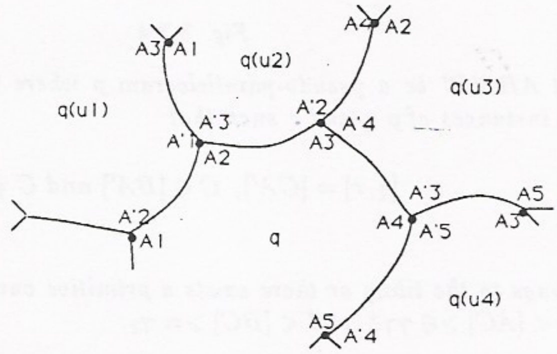

Fig. 5.8

Proof.— First let us observe that $q(u_1), \cdots, q(u_4)$ exist by lemma 4.2. Let $[q, q(u_i)] = [A_i A_{i+1}]$. Then by lemma 4.5, we have

$$[q(u_i), q(u_{i+1}] = [A_i A_{i+1}] \text{ and } [q(u_{i+1}, q(u_i)] = [A'_{i+2} A_i]$$

Assume that $u_1 \neq u_2 - u_3$ and $u_4 \neq u_3 - u_2$. Since $[q, q(u_1)] = [A_1 A_2]$ and $[q(u_3), q(u_2)] = [A'_4 A_2]$ it implies $A_1 \neq A'_4$.

In the same way $u_4 \neq u_3 - u_2$ implies $A_2 \neq A'_5$. Since (A'_2, A'_4, A_3) is a contact, so is (A_2, A_4, A'_3). And we have two distinct contacts (A_2, A'_1, A'_3) and (A_2, A_4, A'_3). We have also two distinct contacts (A'_3, A'_5, A_4) and (A'_3, A_2, A_4). We have $A_1 \neq A_2$ so lemma 5.1 can be applied to the first distinct contacts. And $A_4 \neq A_5$ implies lemma 5.1 can be applied to the last distinct contacts. So we have the scheme:

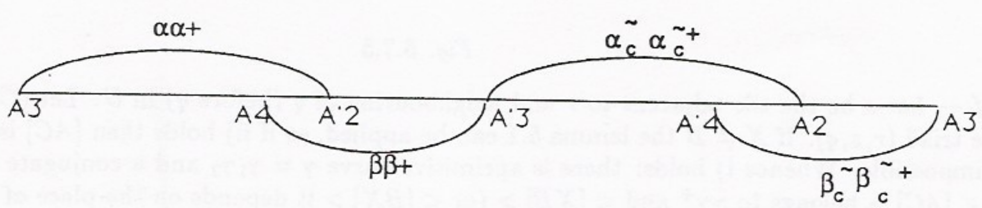

Fig. 5.8bis

But $[q(u_2), q(u_3)]$ is of length l. So $|[A_3 A_4]| \leq |[A_4 A'_2]|$, whence $|\alpha| \leq |[A_4 A'_2]|$.

Also $|[A'_2 A'_3]| \leq |[A_4 A'_2]|$, whence $|\beta| \leq |[A_4 A'_2]|$. it implies that the curve centerd in A'_3 with length $2inf(|\alpha|, |\beta|)$ is a closed curve and there is a contradiction. So i) is proved. Now, ii) is an immediate consequence of Corollary 5.5.2. ■

Definition.— Let e be the edge $[q(u_2), q(u_3)]$.

If $u_1 = u_2 - u_3$ and $u_4 = u_3 - u_2$, we will say that the maximum edge is propagated *two-sidedly* over q (Figure 5.9.1).

If $u_1 = u_2 - u_3$ and $u_4 \neq u_3 - u_2$, the maximum edge e is propagated *on the left* over q (Figure 5.9.2). If $u_1 \neq u_2 - u_3$ and $u_4 = u_2 - u_3$ the edge e is propagated *on the right* over q (Figure 5.9.3).

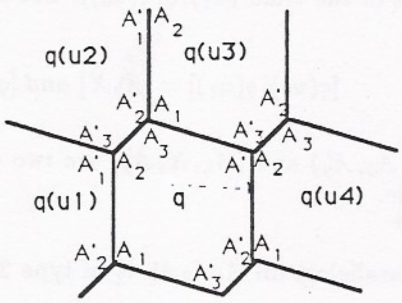

Fig. 5.9.1

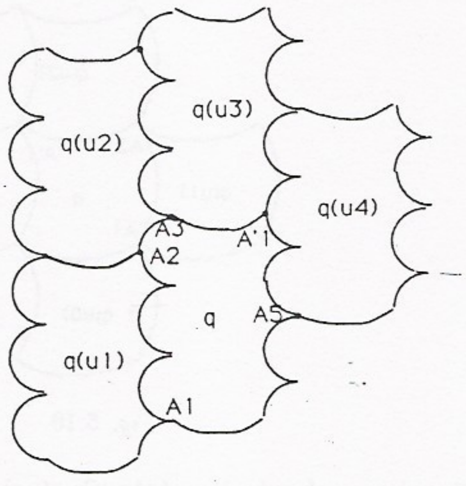

Fig. 5.9.2

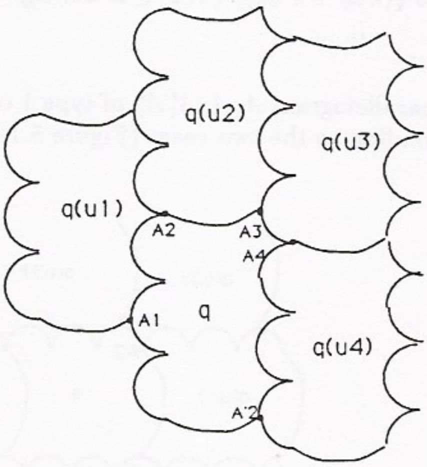

Fig. 5.9.3

At this step, two cases have to be considered.

Case 1 Every time the edge e appears in the tiling, it is two-sidedly propagated *at the two extremities of* e (so, four new occurrences of e appear).

Case 2 There exists an occurrence of e in the tiling, and an extremity of e such that e is propagated only on the left, or on the right.

Case 1

We can use the scheme of Figure 5.3.1. The curve $[A'_2A_1]$ has a strictly positive length. So there exists in U, in a surrounding of q, a tile $q(u_0)$ neighbouring to $q(u_1)$ and adjacent to q (before $q(u_1)$). Let (A'_2, A_1, X) be the contact of the triad $(u_1), q, q(u_0))$. Let us observe that:

$$[q(u_0), q(u_1)] = [A_2X] \textit{ and } [q, q(u_2)] = [A_2A_3]$$

• If $X \neq A_3$ then (A'_1, A_3, A'_2) and (A_1, X, A'_2) are two distinct contacts. So, using lemma 5.1 and 5.2, two situations are available.

• If $X \neq A_2$

Then q is a pseudo-parallelogram $A_1A_2A'_1A'_2$ of type 2 ($A_3 = A'_1$). And we have the following scheme (Figure 5.10):

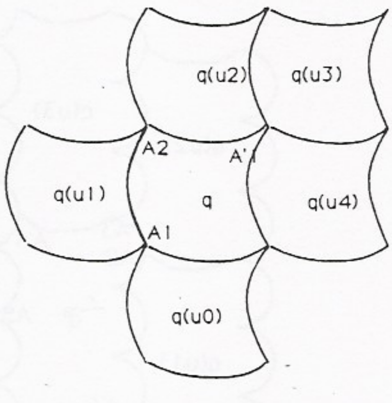

Fig. 5.10

Now considering the edge $e = [q, q(u_4)]$ and the tile $q(u_0)$, this edge is propagated two-sidedly over $q(u_0)$ so $q(u_4 + u_0)$ belongs to U. In the same way, $q(u_1 - u_0)$ and $q(u_1 + u_0)$ belong to U. Iterating this argument, we prove that all the tiles $q(ku_0 + k'u_4)$, $(k, k' \in \mathbf{Z}$ belong to U, but these tiles realize a regular tiling. So, U is regular.

• If $X \neq A_2$

Then q is a pseudo-parallelogram $A_1A_2A'_1A'_2$ of type 1 or of type 2 with $[A_2A'_1]$ a line segment and two possible schemes corresponding to the two cases (Figure 5.11.1, 5.11.2).

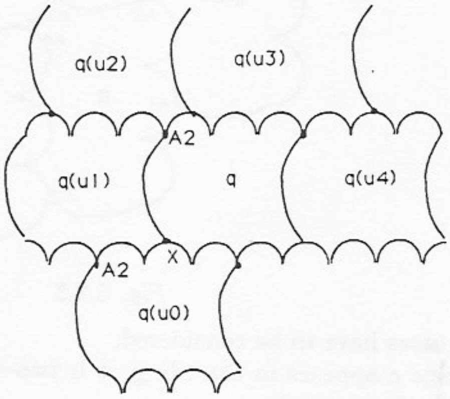

Fig. 5.11.1

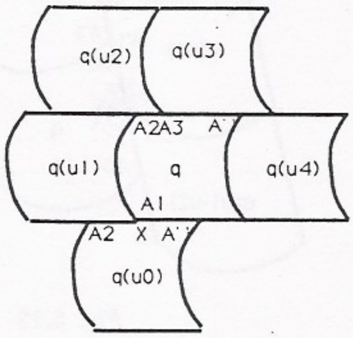

Fig. 5.11.2

In the two cases, we iterate the process of propagation of the edge $[q(u_2), q(u_3)]$ and we obtain that the tiling is invariant in the translation of vector $u_3 - u_2 = u_4$.

- If $X = A_3$ we obtain the Figure 5.11.3

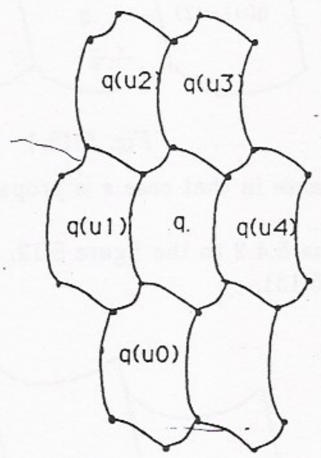

Fig. 5.11.3

By the hypothesis of propagation of the edge $[q(u_1), q]$, $q(u_4 + u_0)$ belongs to the tiling, and iterating the process, we obtain that U is half-periodic for vector $u_4 = u_3 - u_2$ (U is not necessarily regular because $[A_2A_1']$ can be a non-primitive curve).

Case 2

We can use the scheme of Figure 5.9.3. Somewhere in the tiling appears this configuration where $[q(u_2), q(u_3)]$ has length l and $u_1 \neq u_2 - u_3$, $u_4 = u_2 - u_3$.

If we look at the triad $(q(u_1), q(u_2), q)$, it has contact (A_1', A_3', A_2). So, since $A_1' \neq A_4$, there are two different contacts (A_2, A_1', A_3') and (A_2, A_4, A_3'). And, looking at the proof of lemma 5.6 we can conclude that p is a pseudo-parallelogram $A_2A_3A_2'A_3'$ of type 1. And because of the maximality of $[q, q(u_4)]$, we can apply lemma 5.4.1, so $q(u_1 - u_2)$ belongs to the tiling(Figure 5.12) (we have represented $[A_3A_2']$ as a line segment, to simplify the picture but it is only assumed to be a non-primitive curve and $< [A_3'A_2] >$ to be the mirror image of a conjugate of $< [A_3A_2'] >$).

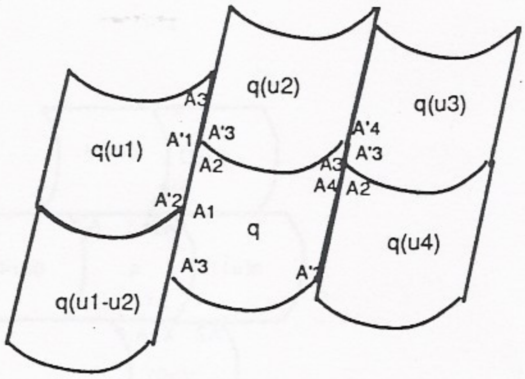

Fig. 5.12

To achieve the reasoning we have to discuss on the place of point A'_1.

- If $A'_1 = A'_2$
  In that case, by maximality of l, we have $A_3 = A_4$. And $u_2 - u_1 = u_3 - u_2 = u_4$ (Figure 5.12.1).

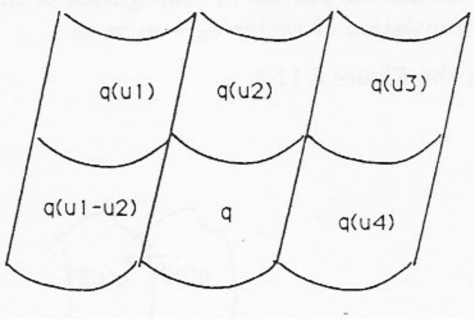

Fig. 5.12.1

There is a contradiction because in that case e is propagated two-sidedly. So, one has:

- $A'_1 \neq A'_2$
  Whence, we can apply lemma 5.4.2 to the figure 5.12. And so $q(-u_2)$ belongs to U. By lemme 5.4.3, $q(u_4 - u_2)$ belongs to U (Figure 5.13).

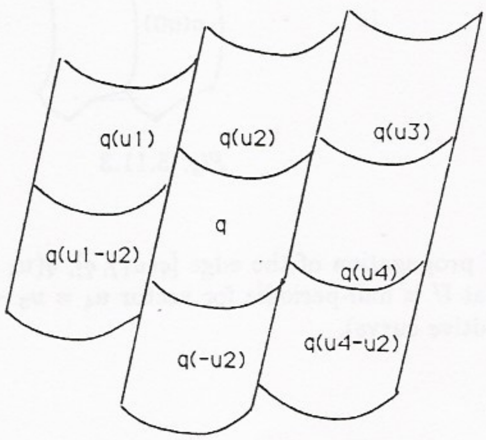

Fig. 5.13

Iterating the process, we obtain that $q(u_1 - ku_2), q(-ku_2), q(u_4 - ku_2)$ belong to U for every $k \geq 0$.

In the same way $q(u_1 + u_2)$ belongs to U by lemma 5.4.1 then $q(u_2)$ belongs to U by lemma 5.4.2, at last $q(u_3 + u_2)$ belongs to U with lemma 5.4.3, and we can iterate the process. So the biinfinite band $B = \{q(ku_2)/k \in \mathbb{Z}\}$ belongs to U.

And the lemma 5.4.4 provides the result in case **A**.

B) Every edge in the tiling has a length strictly less than l.—

It is possible that in the tiling the upperbound of the lengths of the edges is not reached. For example, with a rectangle, we can realize a tiling where the upperbound of lengths of edges is equal to the larger side of the rectangle, but where no edge has this size. So there exists a point A of the boundary of p which is an accumulation point for the left extremity of the edges of the tiling (because l is not attempted, there is an infinite number of edges). Let E be the set of edges of U. From E we can extract an infinite sequence S_1 of edges $S_1 = ([A_nB_n])$ so that their left extremity A_n converges monotonously to A (by the left or by the right) and such that the upperbound of lengths of edges in S_1 is equal to l. Now, looking at the right extremity B_n of edges of S_1 we can extract a subsequence S_2 such that this right extremity converges to a point B, monotonously (by the left or by the right) and keeps again an upperbound of lengths equal to l. So it implies that $|<[AB]>| = l$.

At last, considering in the tiling U, the contacts in points A_n, they can be written (A_n, C_n, B'_n), and one more time, we extract a subsequence S_3 of S_2 so that the sequence C_n converges monotonously to a point C.

The sequence (A_n) and (B_n) cannot be ultimately constant because there is no edge of length l.

But we will prove now the following statement:

Lemma.— *Among the three sequences $(A_n), (B_n), (C_n)$, we have:*

(A_n) and (C_n) are trivial and (B_n) is not trivial or (B_n) and (C_n) are trivial and (A_n) is not trivial.

Proof.— The two sequences (A_n) and (B_n) cannot be both trivial because of the hypothesis about l. Let us suppose for example that (B_n) is trivial but not (A_n). Then A_n converges to A monotonously by the right side. We have to prove that (C_n) is not trivial. If not, (C_n) converges to C by the right side (Figure 5.14); otherwise, since A'_n converges to A on the right we would have a sequence of edges $[C_nA'_n]$ which are nested and it is impossible because of corollary 4.5 bis.

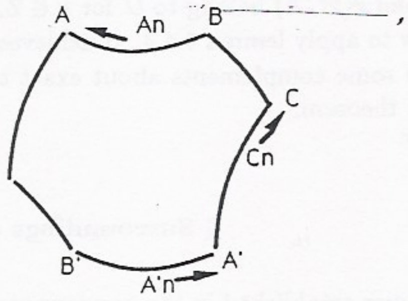

Fig. 5.14

So if (C_n) is not trivial, (C_n) converges to C by the right side and the edges $[C_nA'_n]$ are overlapping.

Let $n > m$. Corollary 4.5 bis provides the following equation:

Let $<[C_n\underline{A'_m}]> = v$.

Since $<[C_mA'_m]> = <[C'_mA_m]>$ and $<[\widetilde{C_nA'_n}]> = <[C'_nA_n]>$, the curve $<C'_nA'_m]>$ admits $\tilde{v}$ as left and right factor. So we have an equation

$$\alpha\tilde{v} = \tilde{v}\beta$$

But n and m cannot be chosen large enough to have $|\alpha| < |v|$. So $\tilde{v}$ belongs to $\alpha^+\alpha'$ where α' is a left factor of α. When n and m grow up, the curve $[C_nA'_m]$ converges to $[CA']$ and $|\alpha|$ converges to zero. It implies that $[CA']$ is a line segment (lemma 3.1).

But if $[CA']$ is a line segment, $[C_nA'_n]$ is also a line segment, for n large enough, and considering three edges $[C_{n_1}A'_{n_1}], [C_{n_2}A'_{n_2}], [C_{n_3}A'_{n_3}]$ with $n_1 < n_2 < n_3$, we get a contradiction:

$[C_{n_3}A'_{n_1}]$ is a line segment containing strictly $[C_{n_2}A'_{n_2}]$. And also $[C'_{n_3}AA_{n_1}]$ is a line segment containing strictly $[C'_{n_2}A_{n_2}]$. So $[C_{n_2}A'_{n_2}]$ cannot be an edge between two instances of p, the edge is necessarily larger.

Finally, we have proved that (C_n) is trivial. And the proof of lemma is achieved. ■

Let us assume that (A_n) and (C_n) are trivial and (B_n) not. Then we have:

Lemma.— *[AC] is a line segment.*

Proof.— By the same kind of argument that in the preceeding lemma we get that [AB] and [BC] are segments, and also $[AB_n]$ and $[B_nC]$ so [AC] is a segment line. So, somewhere in the tiling we have the following situation: a contact (A, C, B'_n) in a triad (q, r, s) (Figure 5.15) and B_n is as near as we want to B. [AC] is a segment line.

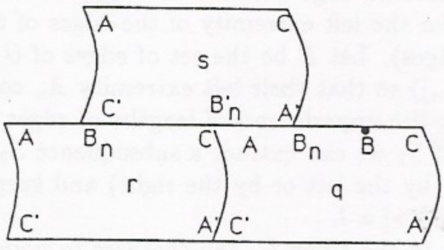

Fig. 5.15

If $s(\overrightarrow{CA})$ does not belong to the tiling, by lemma 5.4.5 it implies that there exists a primitive curve $\gamma_n = \gamma_{1,n}\gamma_{2,n}$ such that

$$< [CA'] >\in \gamma_n^+ \gamma_{1,n} \text{ and } < [A'B'_n] >= \gamma_{2,n}.$$

But this cannot happen for another value $m > n$. Indeed we would have $< [A'B'_m] >= \gamma_{2,m}$. And $|\gamma_{2,m}| > |\gamma_{2,n}|$. So $\gamma_{1,n}$ would end with a line segment oriented as $[A'B'_n]$. Then there is an impossibility for p in point C', the boundary of p would admit a factor centered in C' which is a closed curve. So changing the value of n $s(\overrightarrow{CA})$ belongs to the tiling. For the same reason, $r(\overrightarrow{CA})$ belongs to U. And now, iterating the process $s(k\overrightarrow{CA})$ and $r(k\overrightarrow{CA})$ belong to U for $k \in \mathbb{Z}$.

We have just now to apply lemma 5.4.4. It achieves the proof of case B and theorem 5.5 is proved. ■

We can give now some complements about exact tiles and their surroundings which are deduced from the proof of the main theorem.

6 Surroundings of exact tiles

The different lemmas established in the previous section are now very useful to describe all the complete surroundings of an exact tile.

Theorem 6.1.— *Every complete surrounding of an exact tile contains* 6, 7, *or* 8*tiles, and the minimal surrounding extracted of the complete one contains respectively* 6, 5, *or* 4 *tiles.*

Proof.— First of all, by Corollary 4.3 every surrounding can be extended in a tiling of the whole plane, so we have just to look at the surroundings appearing in the tilings of the plane, and observe in the proof of theorem 5.5 what kinds of surrounding appear.

We give below the different complete and respective minimal surroundings of an exact tile.

Complete 6-surroundings.—

These surroundings are also minimal.

*$< BA' >=< B'A >$

* There exists a curve $\alpha = \alpha_1\alpha_2$ such that:

$$< AC >\in \alpha^*\alpha_1, \quad < CB >\in \alpha_2\alpha^*$$

$$< A'D >\in \tilde{\alpha_1}\tilde{\alpha^*}, \quad < DB' >\in \tilde{\alpha^*}\tilde{\alpha_2}$$

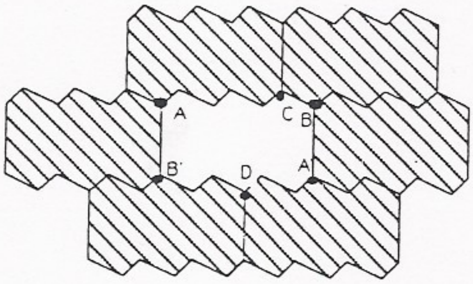

Fig. 6.1

Complete 7-surroundings.—

The associated minimal ones contain 5 tiles.

$* < BA' > = < \widetilde{B'A} >$

*There exists a curve a such that:

$$< AB > \in \alpha\alpha^+, \quad < A'C > \in \widetilde{\alpha^+}, \quad < CB' > \in \widetilde{\alpha^+}$$

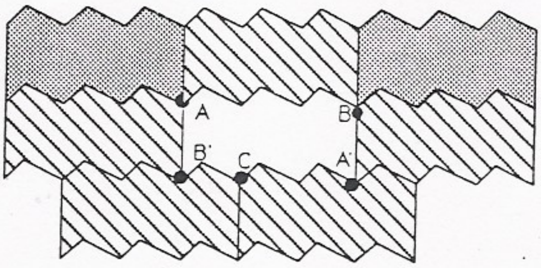

Fig. 6.2

Complete 8-surroundings.—

The associated minimal ones contain 4 tiles.

$* < AB > = < \widetilde{A'B'} >, \; < BA' > = < \widetilde{B'A} >$

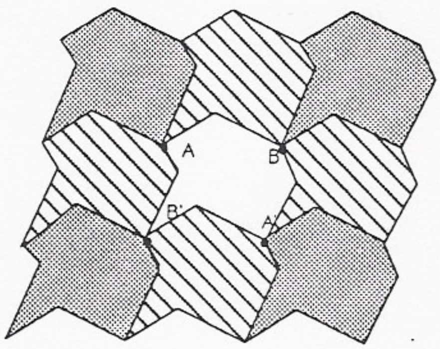

Fig. 6.3

From this description we deduce immediately the following property:

Proposition 6.2.— *If a tile has a complete surrounding with 7 tiles then it has also a complete surrounding with 6 or 8 tiles.*

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