Consistency of combinatorial substitutions

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Séminaire Combinatoire et Algorithmes
LITIS, Université de Rouen
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Substitutions: replace letters by patterns.

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Examples. . .

$$1 \mapsto \begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix} \qquad 2 \mapsto \begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}$$

1

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$$1 \mapsto \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \mapsto \begin{matrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{matrix} \mapsto \begin{matrix} 1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 & 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 & 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 \end{matrix}$$

$$1 \mapsto \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \qquad\qquad 2 \mapsto \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}$$

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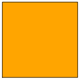

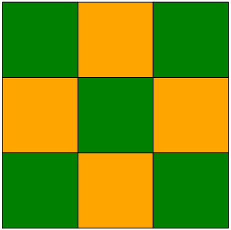

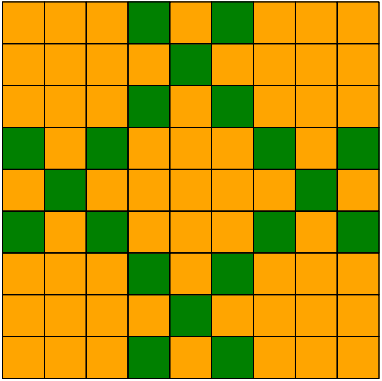

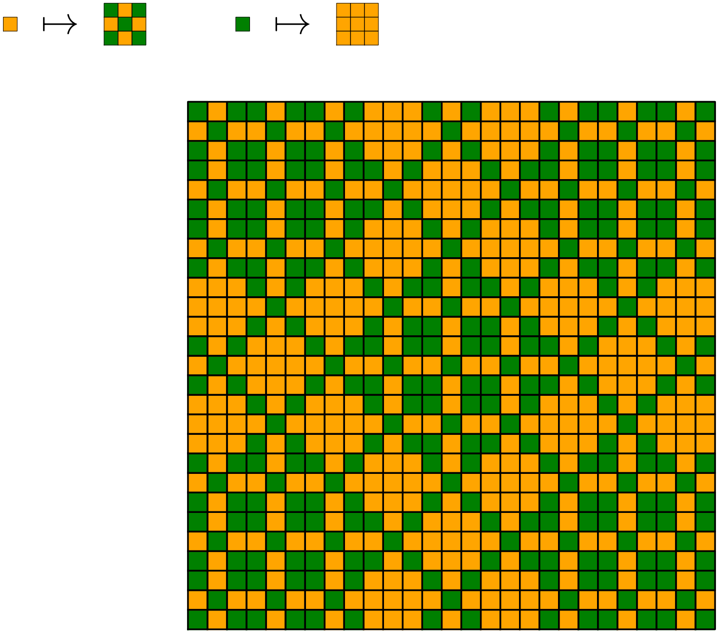

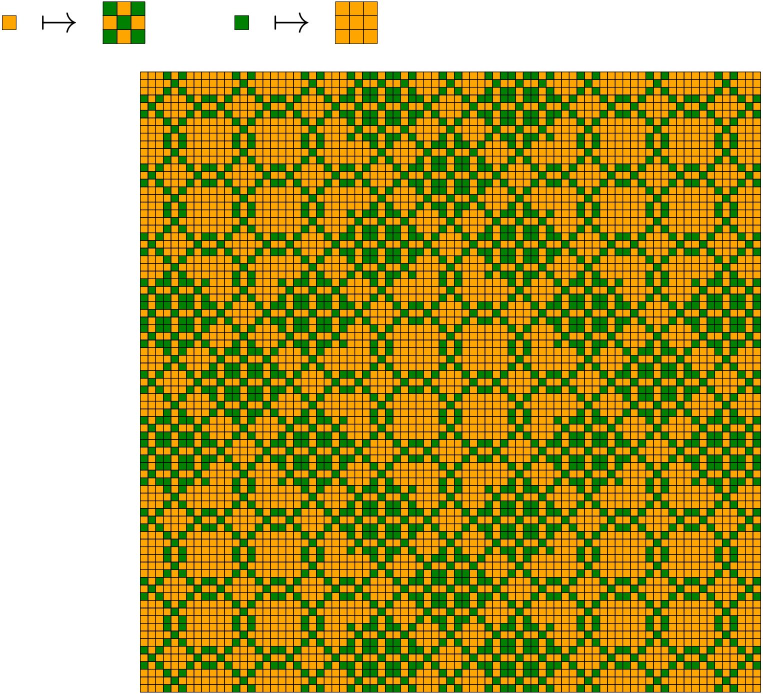

$$\boxed{1} \;\longmapsto\; \begin{array}{|cc|}\hline 2 & 4 \\ 1 & 3 \\ \hline\end{array} \qquad \boxed{2} \;\longmapsto\; \boxed{1\;\;3} \qquad \boxed{3} \;\longmapsto\; \begin{array}{|c|}\hline 2 \\ 1 \\ \hline\end{array} \qquad \boxed{4} \;\longmapsto\; \boxed{1}$$

$$\boxed{1}$$

$$\boxed{1} \longmapsto \boxed{\begin{matrix} 2 & 4 \\ 1 & 3 \end{matrix}} \qquad \boxed{2} \longmapsto \boxed{1 \ \ 3} \qquad \boxed{3} \longmapsto \boxed{\begin{matrix} 2 \\ 1 \end{matrix}} \qquad \boxed{4} \longmapsto \boxed{1}$$

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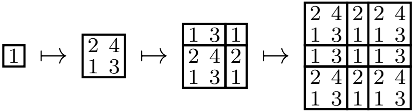
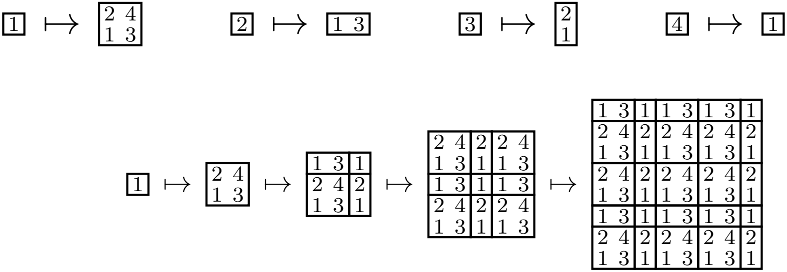
$$\boxed{1} \mapsto \boxed{\begin{array}{cc} 2 & 4 \\ 1 & 3 \end{array}} \mapsto \begin{array}{|cc|c|} \hline 1 & 3 & 1 \\ \hline 2 & 4 & 2 \\ 1 & 3 & 1 \\ \hline \end{array}$$





"Compatibles" rectangles.

11 letters, compatible rectangles (from M. Rigo)

$$a \mapsto \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \quad b \mapsto \begin{array}{|c|}\hline i \\ \hline e \\ \hline\end{array} \quad c \mapsto \begin{array}{|c|c|}\hline i & j \\ \hline\end{array} \quad d \mapsto \begin{array}{|c|}\hline i \\ \hline\end{array} \quad e \mapsto \begin{array}{|c|c|}\hline f & b \\ \hline\end{array}$$

$$f \mapsto \begin{array}{|c|c|}\hline g & b \\ \hline h & d \\ \hline\end{array} \quad g \mapsto \begin{array}{|c|c|}\hline f & b \\ \hline h & d \\ \hline\end{array} \quad h \mapsto \begin{array}{|c|c|}\hline i & m \\ \hline\end{array} \quad i \mapsto \begin{array}{|c|c|}\hline i & m \\ \hline h & d \\ \hline\end{array}$$

$$j \mapsto \begin{array}{|c|}\hline k \\ \hline c \\ \hline\end{array} \quad k \mapsto \begin{array}{|c|c|}\hline l & m \\ \hline c & d \\ \hline\end{array} \quad l \mapsto \begin{array}{|c|c|}\hline k & m \\ \hline c & d \\ \hline\end{array} \quad m \mapsto \begin{array}{|c|}\hline i \\ \hline h \\ \hline\end{array}$$

26 letters, compatible rectangles (from M. Rigo)

$$a \mapsto \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \quad b \mapsto \begin{array}{|c|}\hline e \\ \hline f \\ \hline\end{array} \quad c \mapsto \begin{array}{|c|c|}\hline e & h \\ \hline\end{array} \quad d \mapsto \begin{array}{|c|}\hline i \\ \hline\end{array} \quad e \mapsto \begin{array}{|c|c|}\hline j & k \\ \hline l & m \\ \hline\end{array}$$

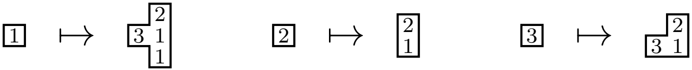
$$f \mapsto \begin{array}{|c|c|}\hline g & b \\ \hline\end{array} \quad g \mapsto \begin{array}{|c|c|}\hline y & b \\ \hline o & t \\ \hline\end{array} \quad h \mapsto \begin{array}{|c|}\hline z \\ \hline c \\ \hline\end{array} \quad i \mapsto \begin{array}{|c|c|}\hline i & n \\ \hline o & d \\ \hline\end{array}$$

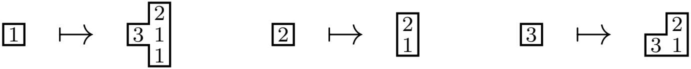
$$j \mapsto \begin{array}{|c|c|}\hline e & p \\ \hline q & r \\ \hline\end{array} \quad k \mapsto \begin{array}{|c|}\hline e \\ \hline s \\ \hline\end{array} \quad l \mapsto \begin{array}{|c|c|}\hline e & u \\ \hline\end{array} \quad m \mapsto \begin{array}{|c|}\hline e \\ \hline\end{array}$$

$$n \mapsto \begin{array}{|c|}\hline i \\ \hline o \\ \hline\end{array} \quad o \mapsto \begin{array}{|c|c|}\hline i & n \\ \hline\end{array} \quad p \mapsto \begin{array}{|c|}\hline e \\ \hline q \\ \hline\end{array} \quad q \mapsto \begin{array}{|c|c|}\hline e & p \\ \hline\end{array} \quad r \mapsto \begin{array}{|c|}\hline e \\ \hline\end{array}$$

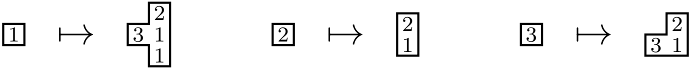
$$s \mapsto \begin{array}{|c|c|}\hline v & k \\ \hline\end{array} \quad t \mapsto \begin{array}{|c|}\hline i \\ \hline\end{array} \quad u \mapsto \begin{array}{|c|}\hline w \\ \hline l \\ \hline\end{array} \quad v \mapsto \begin{array}{|c|c|}\hline w & p \\ \hline l & r \\ \hline\end{array} \quad w \mapsto \begin{array}{|c|c|}\hline v & k \\ \hline q & r \\ \hline\end{array}$$

$$x \mapsto \begin{array}{|c|c|}\hline z & n \\ \hline c & d \\ \hline\end{array} \quad y \mapsto \begin{array}{|c|c|}\hline g & b \\ \hline o & d \\ \hline\end{array} \quad z \mapsto \begin{array}{|c|c|}\hline x & n \\ \hline c & t \\ \hline\end{array}$$



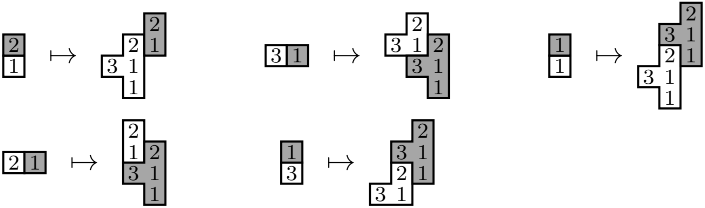


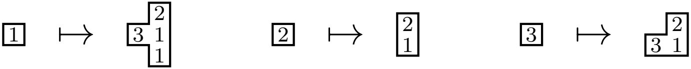
How can we “concatenate” these ?



How can we “concatenate” these ?

➥ We must define some **rules:**





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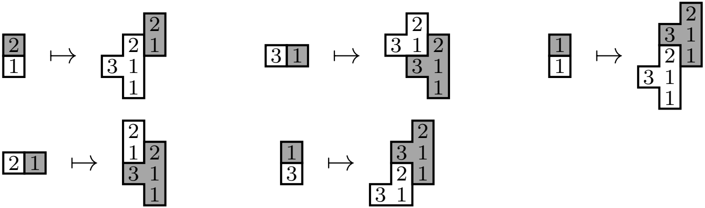

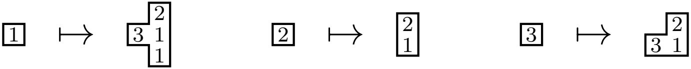

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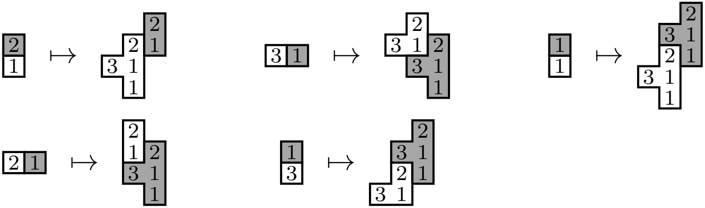

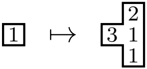

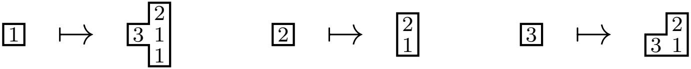

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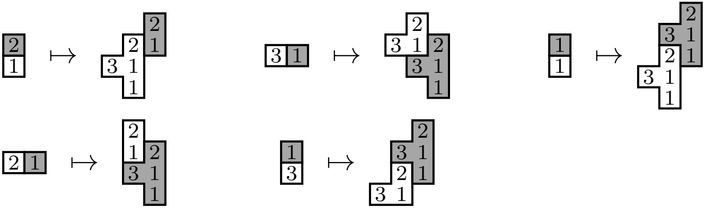

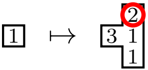

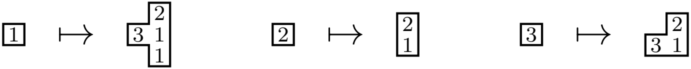

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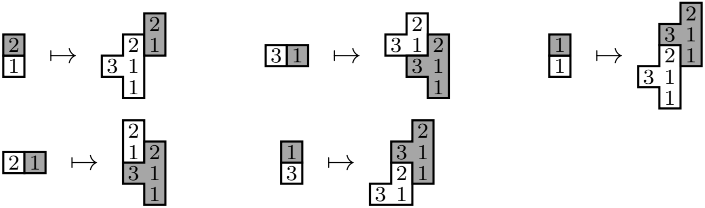

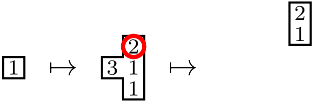

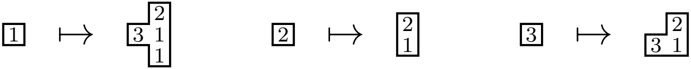

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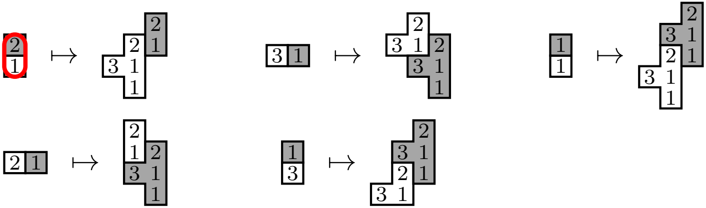

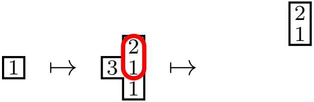

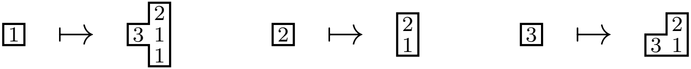

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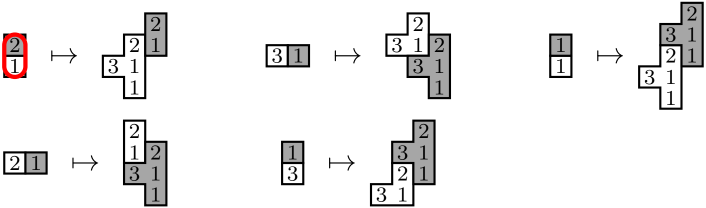

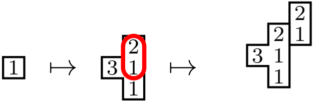

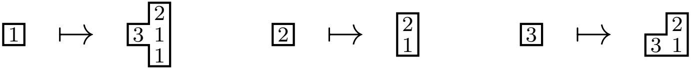

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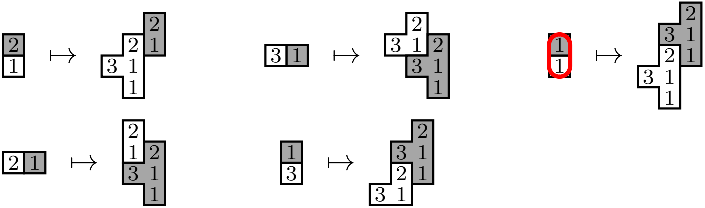

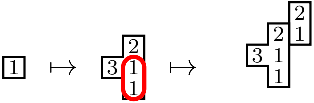

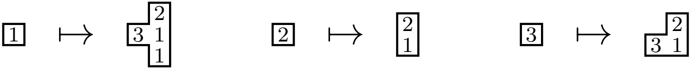

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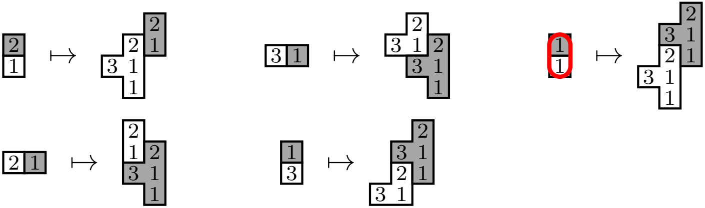

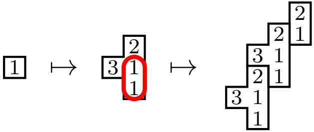

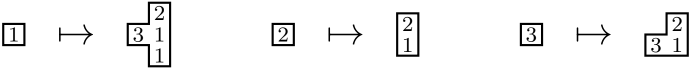

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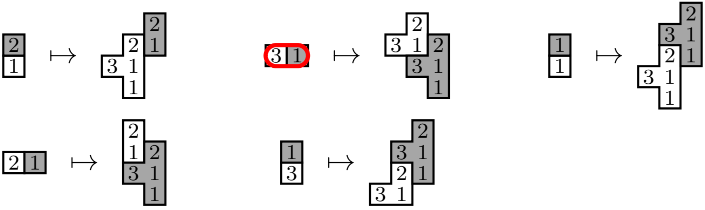

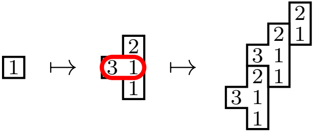

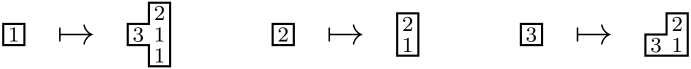

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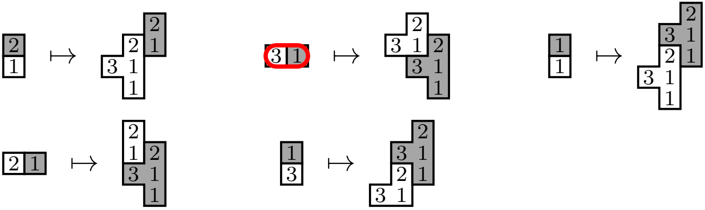

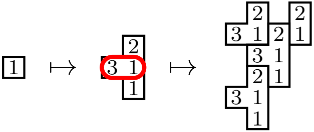

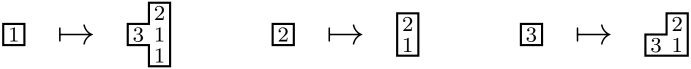

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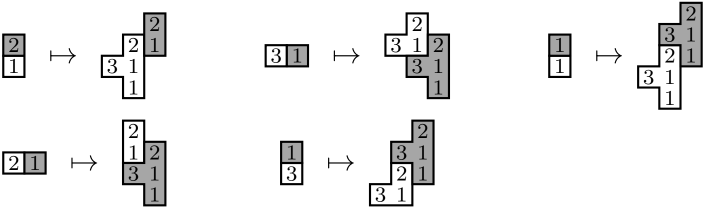
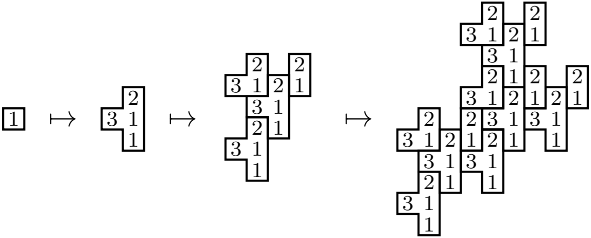
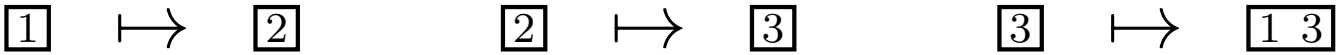
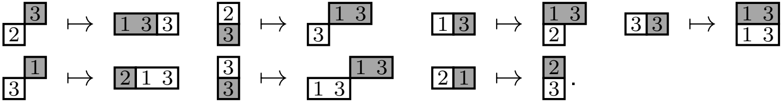
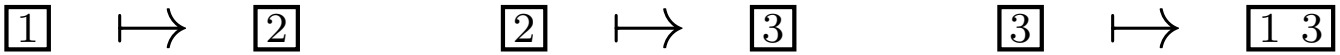






Concatenation rules:





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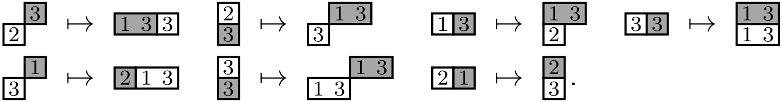

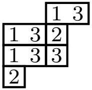

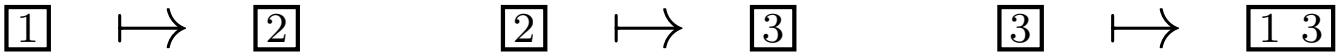

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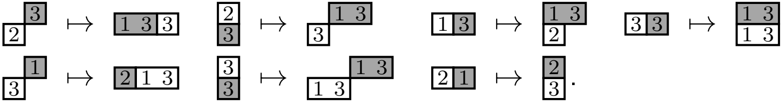

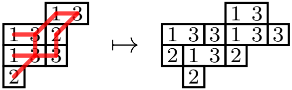

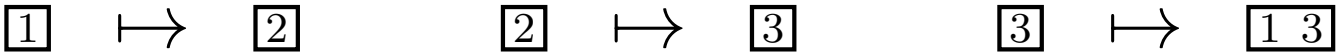

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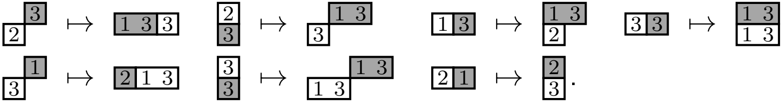

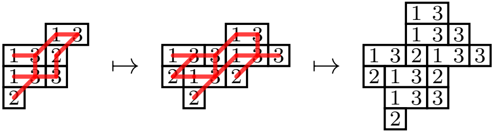

1 ↦ 2   2 ↦ 3   3 ↦ 1 3

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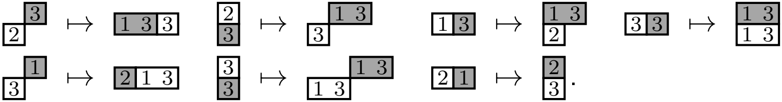

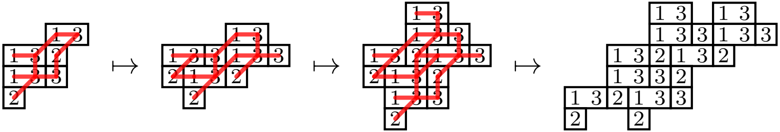

$\boxed{1} \mapsto \boxed{2}$ $\quad$ $\boxed{2} \mapsto \boxed{3}$ $\quad$ $\boxed{3} \mapsto \boxed{1\ 3}$

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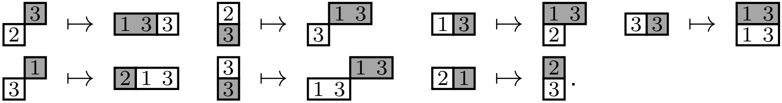
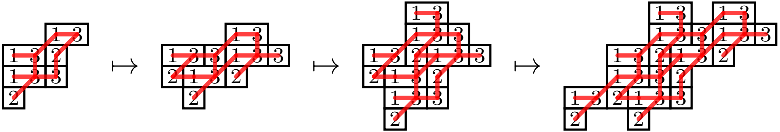
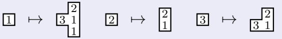
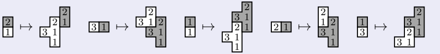




Definition: a **combinatorial substitution** is given by:

1. base rules (),
2. concatenation rules ().

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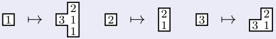
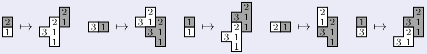
1. base rules (),
2. concatenation rules ().

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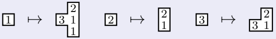
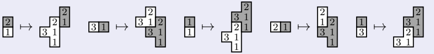
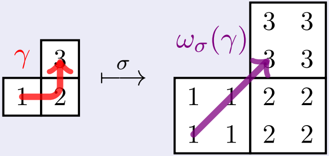
1. base rules (),
2. concatenation rules ().

- Nice and very general definition. . .
- **But:**
  - Such substitutions are not necessarily consistent.
  - Such substitutions can produce overlapping patterns.

Consistency

Notation: if γ is a σ-path, let $\omega_\sigma(\gamma)$ be the **image vector** of γ.



Meaning:
$\sigma(1)$ and $\sigma(3)$
separated by $\omega_\sigma(\gamma)$

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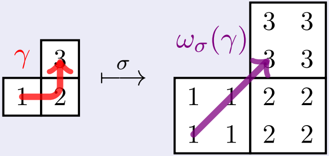

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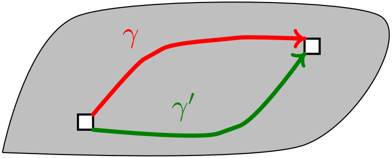

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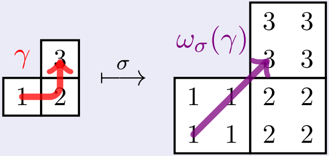

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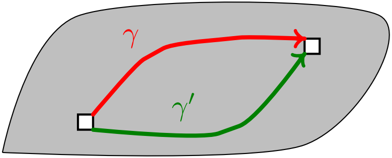

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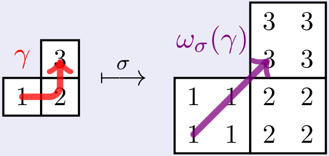

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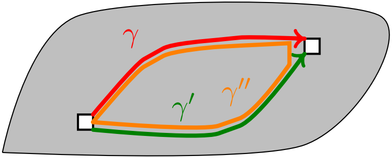

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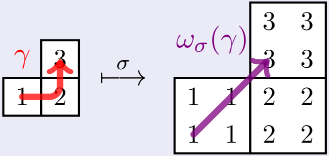

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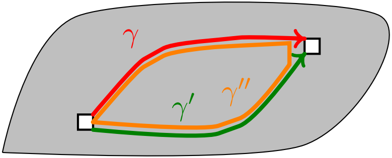

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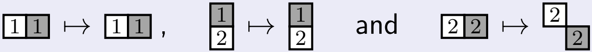
Property: σ is consistent $\iff$ $\omega_\sigma(\gamma) = 0$ for all loops γ.

Example: inconsistent substitution

- Base: $\boxed{1} \mapsto \boxed{1}$ and $\boxed{2} \mapsto \boxed{2}$
- Concatenation rules:



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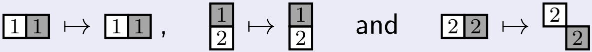

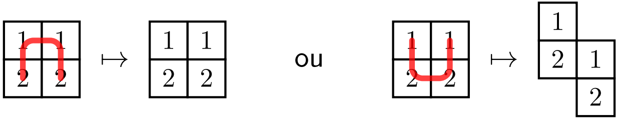

The pattern has two different images, depending on the chosen rules.

Example: overlapping substitution

- Base: $\boxed{1} \mapsto \boxed{1}$
- Concatenation rules: $\boxed{1}\boxed{1} \mapsto \boxed{1}\boxed{1}$ and $\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline \end{array} \mapsto \boxed{1}\boxed{1}$

Example: overlapping substitution

- Base: $\boxed{1} \mapsto \boxed{1}$
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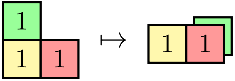

Patterns overlap in the image.

Can **inconsistency** and **overlapping** be detected algorithmically ?

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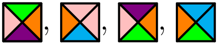
➥ **No.**

Proof methods: tiling problems.

Wang tiles

- **Wang tiles** $=$ unit squares with colored edges.
  Example: $T = \Big\{$  $\Big\}$.

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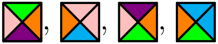
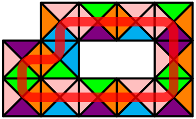
  Example: $T = \{$  $\}$.
- **Valid cycle of T:** sequence of tiles $(a_1, \ldots, a_n)$ such that
  1. a_i is a translated copy of a tile of T (no rotations);
  2. a_i and a_{i+1} share one edge and agree in color on it;
  3. $a_1 = a_n$.



Input: a set T of Wang tiles.
Question: Does there exist a valid cycle of T?

➥ Undecidable [Kari 2002].
➥ Let's use it to prove the undecidability of consistency.

Undecidability of consistency

The reduction: $T \ \mapsto \ \sigma_T$

- Alphabet: $T \times \{\rightarrow, \uparrow, \leftarrow, \downarrow\}$

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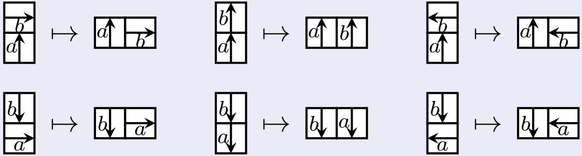
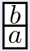
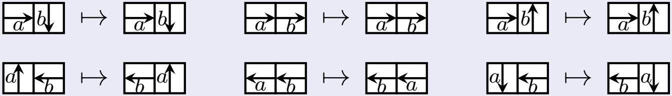
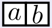
- Concatenation rules:



for tiles $a, b \in T$ such that  match in color, and



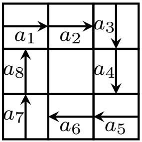
for $a, b \in T$ such that  match in color.

Undecidability of consistency

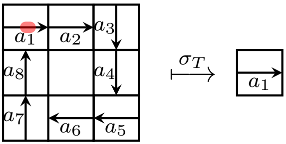
What does σ_T do?

➥ The pointed tile is placed at the right.

Undecidability of consistency



Undecidability of consistency



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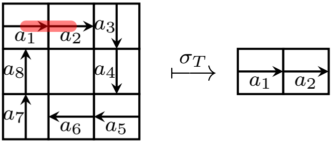

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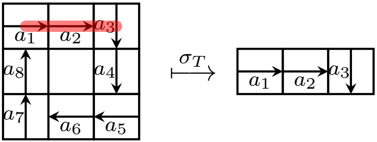

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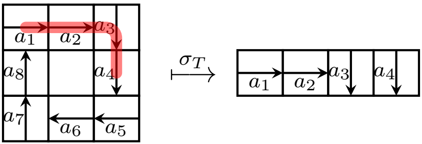

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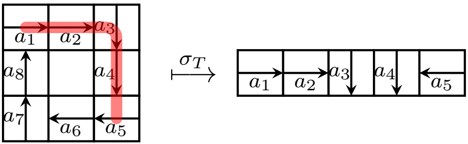

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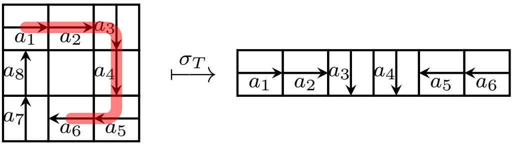

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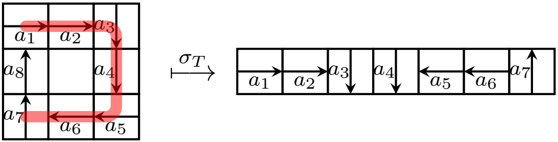

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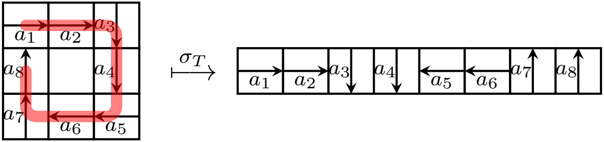

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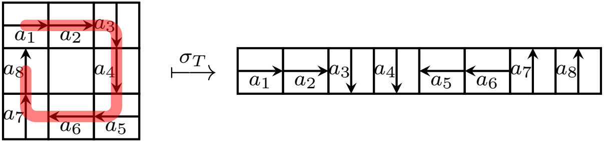

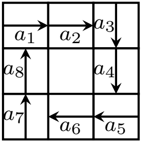

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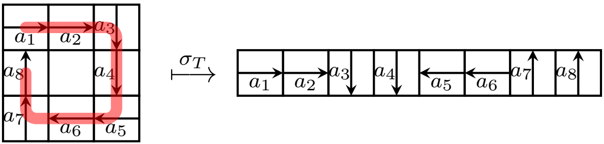

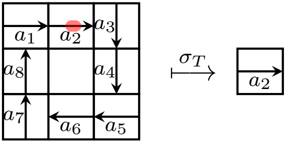

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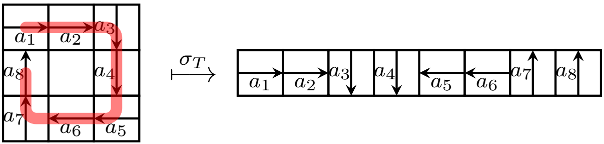

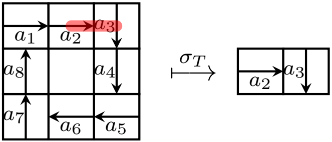

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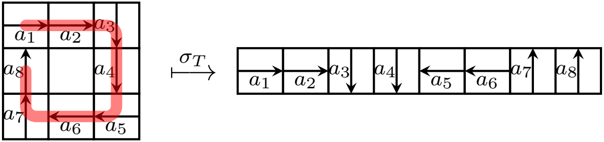

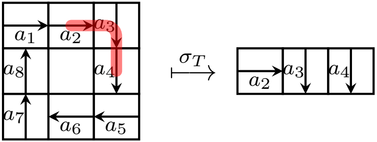

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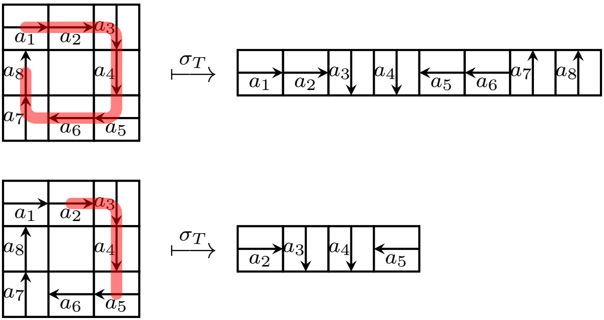

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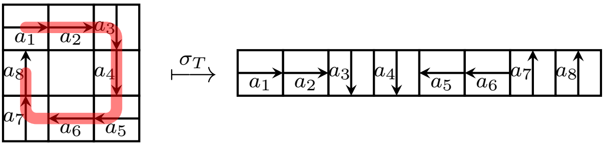

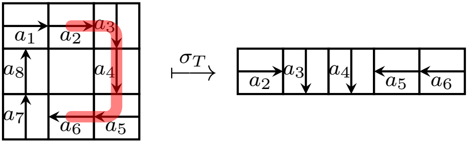

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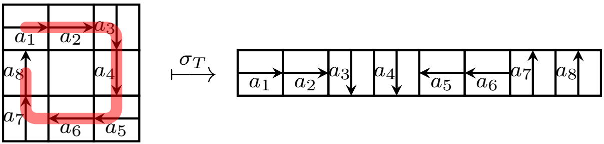

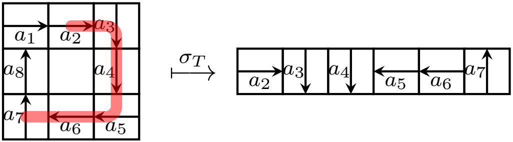

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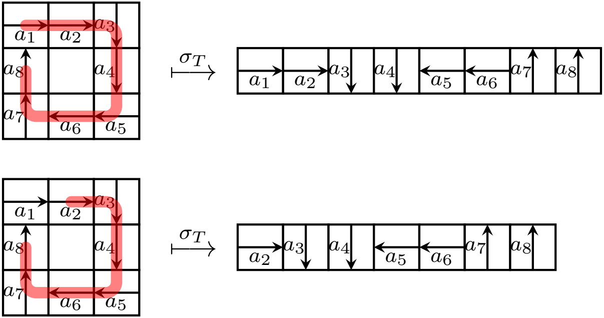

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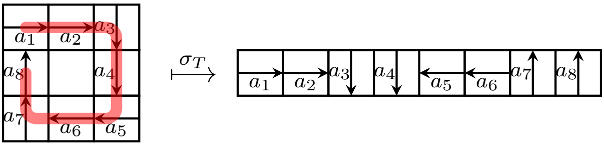
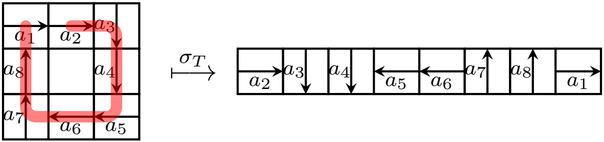
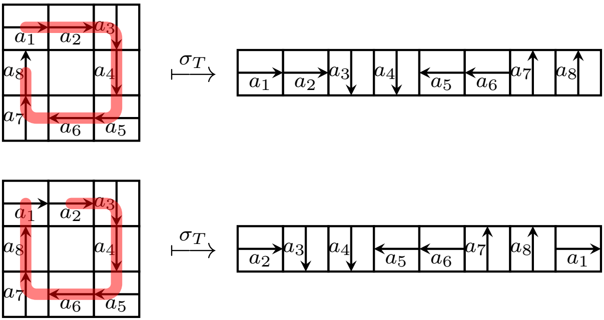




Undecidability of consistency



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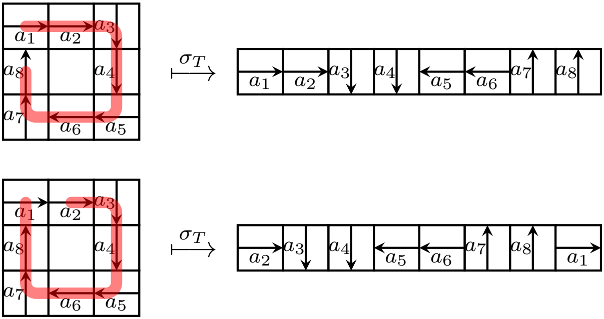

- **Hence:** $\exists$ a valid cycle of $T \implies \sigma_T$ is inconsistent
- (**Conversely:** T is inconsistent
  $\implies$ $\exists$ a simple loop γ
  $\implies$ $\exists$ a valid cycle of T)

➥ **The reduction is complete.** □

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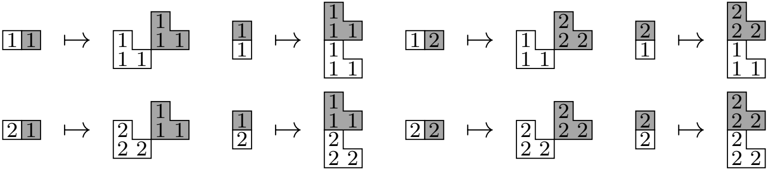
➥ Holds for "reasonnable" classes of substitutions ("domino-to-domino" rules).

Particular case: domino-complete substitutions

Definition

σ is **domino-complete** if there is one rule for every possible domino.

Example:



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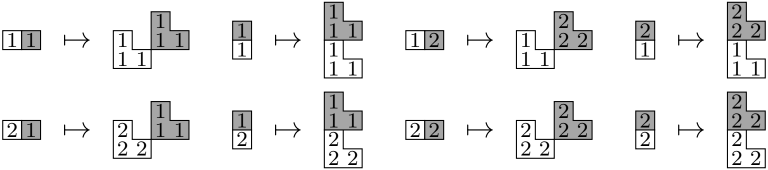

We can **decide consistency** for this class.

Domino-complete substitutions: consistency

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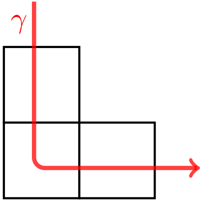

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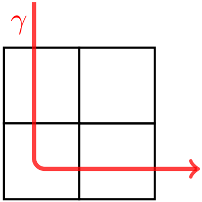

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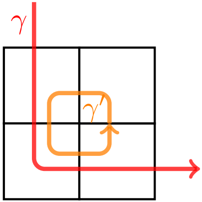

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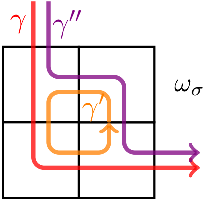

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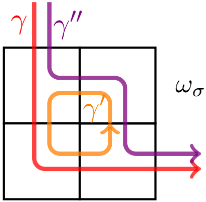

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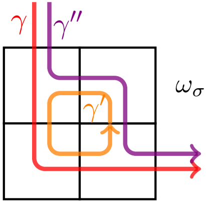

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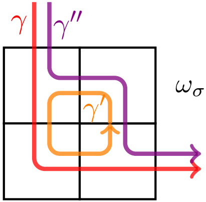

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Theorem [J.-Kari 2011]

A domino-complete substitution is consistent
if and only if it is on the 2×2 patterns: **decidable**.

What about overlapping?

We also have:

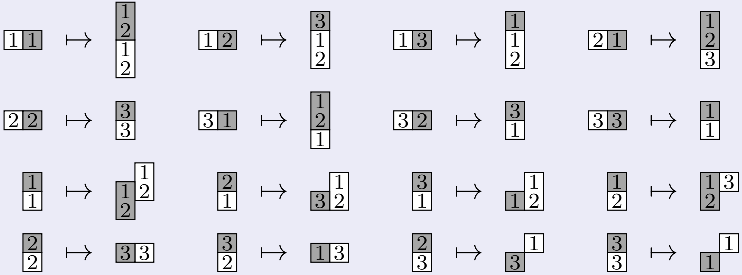
Theorem [J.-Kari 2011]

It is decidable if a consistent domino-complete substitution is overlapping.

Now we study a particular example σ:

Base: $\boxed{1} \mapsto \boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}$, $\quad \boxed{2} \mapsto \boxed{3}$, $\quad \boxed{3} \mapsto \boxed{1}$

Concatenation rules:



➥ "Almost" domino-complete: only $\boxed{2\;3}$ and $\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}$ are omitted.

Iterations of σ

$\boxed{1}$

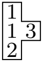
Iterations of σ

1
2

Iterations of σ

1	3
2	

Iterations of σ



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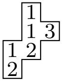

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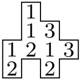

Iterations of σ

		1	
		1	3
	1	2	
	1	3	
1	2	1	3
2		2	

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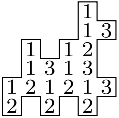

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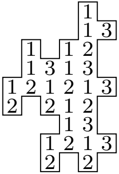

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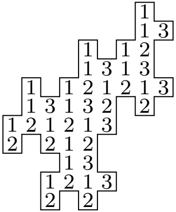

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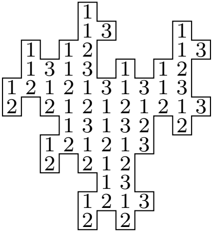

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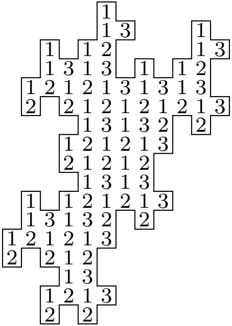

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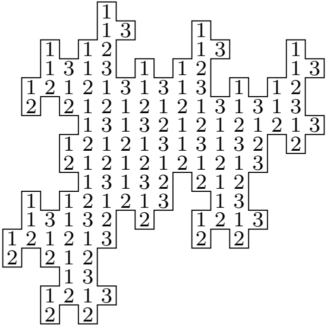

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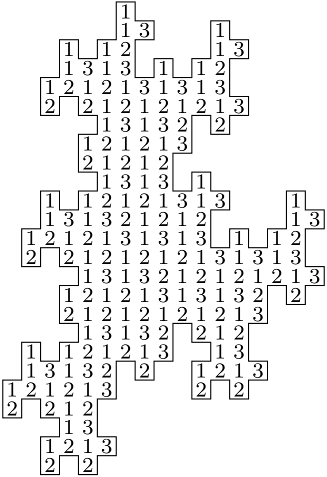

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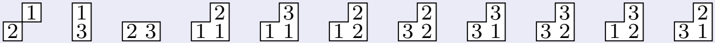

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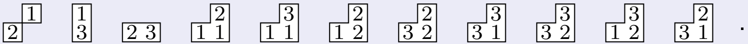
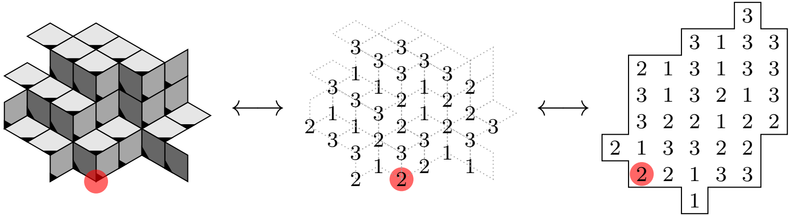


➥ **What?!**

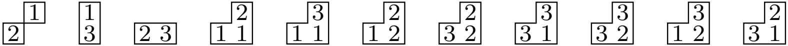
History. . .

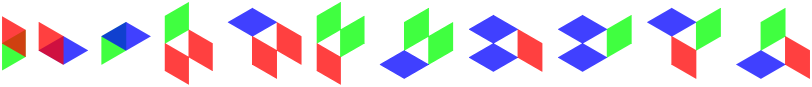
σ is strongly linked with discrete surfaces.
(Ito-Ohtsuki, Arnoux, Berthé, Siegel, Jamet, Fernique,. . .)

Discrete surfaces $\Longleftrightarrow$ configurations of $\{1,2,3\}^{\mathbb{Z}^2}$:



With 11 forbidden patterns:





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Non-overlapping ? : not so easy because of the complicated “algebra” of discrete surface σ-paths.

Questions and problems

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Thank you