

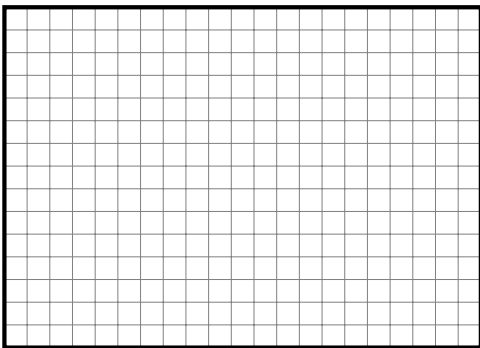
# Journées du GDR IM 2014

**Timo Jolivet**

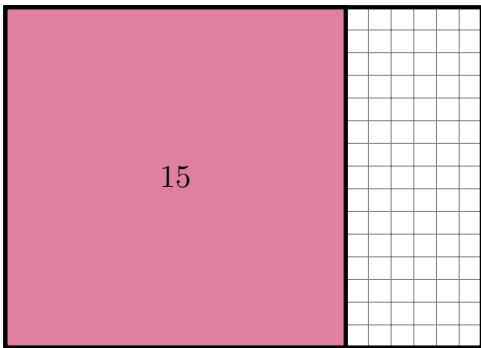
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Univ Turku, Finlande

Post-doctorat :  
Univ Leoben, Autriche  
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2014-01-29

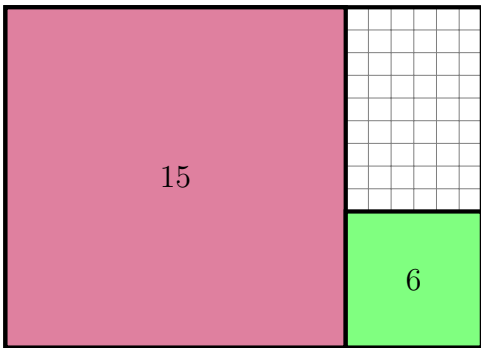


(15, 21)



$(15, 21)$

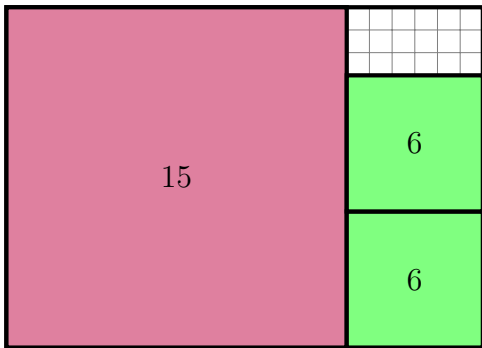
$(6, 15)$



(15, 21)

(6, 15)

(6, 9)

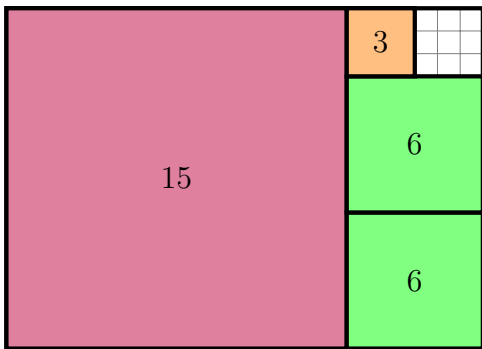


(15, 21)

(6, 15)

(6, 9)

(3, 6)



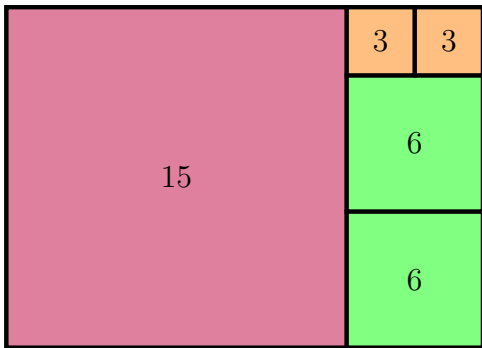
(15, 21)

(6, 15)

(6, 9)

(3, 6)

(3, 3)



(15, 21)

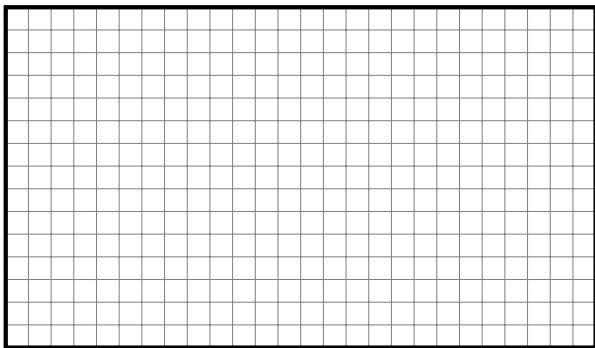
(6, 15)

(6, 9)

(3, 6)

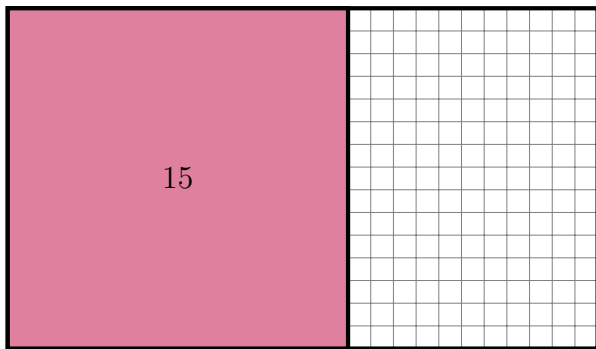
(3, 3)

(0, **3**)



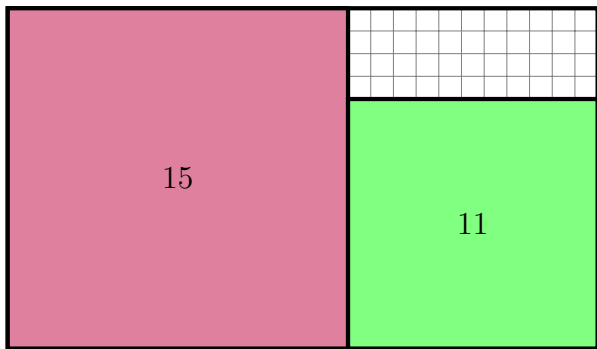
(15, 26)





(15, 26)

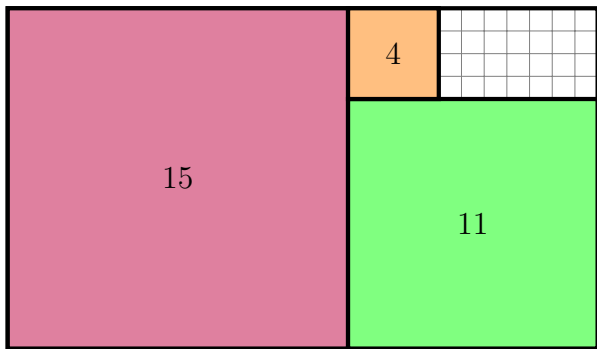
(11, 15)



(15, 26)

(11, 15)

(4, 11)

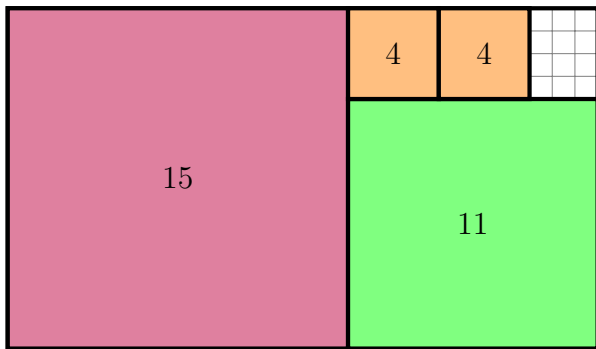


(15, 26)

(11, 15)

(4, 11)

(4, 7)



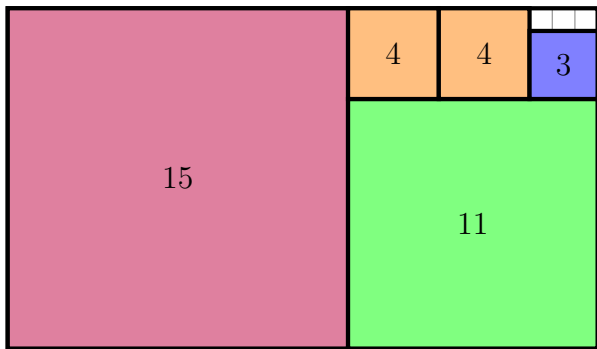
(15, 26)

(11, 15)

(4, 11)

(4, 7)

(3, 4)



(15, 26)

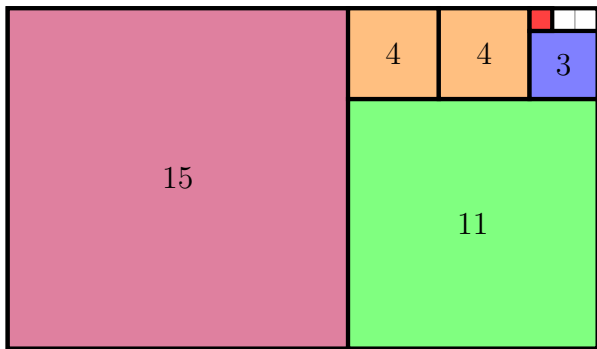
(11, 15)

(4, 11)

(4, 7)

(3, 4)

(1, 3)



(15, 26)

(11, 15)

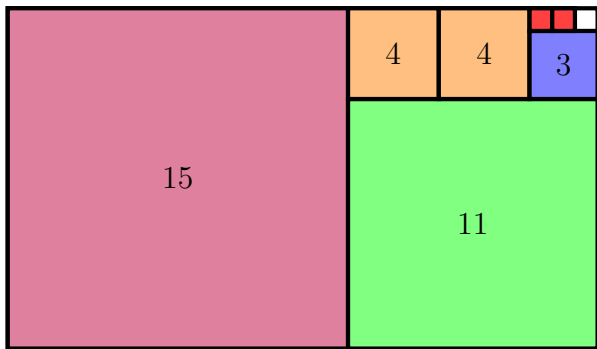
(4, 11)

(4, 7)

(3, 4)

(1, 3)

(1, 2)



(15, 26)

(11, 15)

(4, 11)

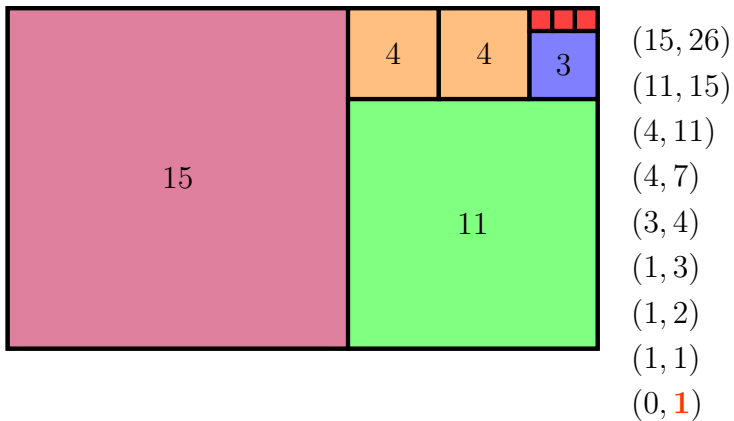
(4, 7)

(3, 4)

(1, 3)

(1, 2)

(1, 1)



**Algorithme d'Euclide : PGCD( $x, y$ )**



$$(1, \sqrt{11})$$

$$(1, \sqrt{11})$$

$$(1, \sqrt{11} - 1)$$

$$(1, \sqrt{11})$$

$$(1, \sqrt{11} - 1)$$

$$(1, \sqrt{11} - 2)$$

$$(1, \sqrt{11})$$

$$(1, \sqrt{11} - 1)$$

$$(1, \sqrt{11} - 2)$$

$$(\sqrt{11} - 3, 1)$$

$$(1, \sqrt{11})$$

$$(1, \sqrt{11} - 1)$$

$$(1, \sqrt{11} - 2)$$

$$(\sqrt{11} - 3, 1)$$

$$(\sqrt{11} - 3, -\sqrt{11} + 4)$$

$$(1, \sqrt{11})$$

$$(1, \sqrt{11} - 1)$$

$$(1, \sqrt{11} - 2)$$

$$(\sqrt{11} - 3, 1)$$

$$(\sqrt{11} - 3, -\sqrt{11} + 4)$$

$$(\sqrt{11} - 3, -2\sqrt{11} + 7)$$

$$(1, \sqrt{11})$$

$$(1, \sqrt{11} - 1)$$

$$(1, \sqrt{11} - 2)$$

$$(\sqrt{11} - 3, 1)$$

$$(\sqrt{11} - 3, -\sqrt{11} + 4)$$

$$(\sqrt{11} - 3, -2\sqrt{11} + 7)$$

$$(-3\sqrt{11} + 10, \sqrt{11} - 3)$$

$(1, \sqrt{11})$   
 $(1, \sqrt{11} - 1)$   
 $(1, \sqrt{11} - 2)$   
 $(\sqrt{11} - 3, 1)$   
 $(\sqrt{11} - 3, -\sqrt{11} + 4)$   
 $(\sqrt{11} - 3, -2\sqrt{11} + 7)$   
 $(-3\sqrt{11} + 10, \sqrt{11} - 3)$   
 $(-3\sqrt{11} + 10, 4\sqrt{11} - 13)$   
 $(-3\sqrt{11} + 10, 7\sqrt{11} - 23)$   
 $(-3\sqrt{11} + 10, 10\sqrt{11} - 33)$   
 $(-3\sqrt{11} + 10, 13\sqrt{11} - 43)$   
 $(-3\sqrt{11} + 10, 16\sqrt{11} - 53)$   
 $(19\sqrt{11} - 63, -3\sqrt{11} + 10)$

...



$A \downarrow$

$$\begin{aligned} & (1, \sqrt{11}) \\ & (1, \sqrt{11} - 1) \\ & (1, \sqrt{11} - 2) \\ & (\sqrt{11} - 3, 1) \\ & (\sqrt{11} - 3, -\sqrt{11} + 4) \\ & (\sqrt{11} - 3, -2\sqrt{11} + 7) \\ & (-3\sqrt{11} + 10, \sqrt{11} - 3) \\ & (-3\sqrt{11} + 10, 4\sqrt{11} - 13) \\ & (-3\sqrt{11} + 10, 7\sqrt{11} - 23) \\ & (-3\sqrt{11} + 10, 10\sqrt{11} - 33) \\ & (-3\sqrt{11} + 10, 13\sqrt{11} - 43) \\ & (-3\sqrt{11} + 10, 16\sqrt{11} - 53) \\ & (19\sqrt{11} - 63, -3\sqrt{11} + 10) \end{aligned}$$

...

$$\begin{aligned} A \downarrow & (1, \sqrt{11}) \\ A \downarrow & (1, \sqrt{11} - 1) \\ & (1, \sqrt{11} - 2) \\ & (\sqrt{11} - 3, 1) \\ & (\sqrt{11} - 3, -\sqrt{11} + 4) \\ & (\sqrt{11} - 3, -2\sqrt{11} + 7) \\ & (-3\sqrt{11} + 10, \sqrt{11} - 3) \\ & (-3\sqrt{11} + 10, 4\sqrt{11} - 13) \\ & (-3\sqrt{11} + 10, 7\sqrt{11} - 23) \\ & (-3\sqrt{11} + 10, 10\sqrt{11} - 33) \\ & (-3\sqrt{11} + 10, 13\sqrt{11} - 43) \\ & (-3\sqrt{11} + 10, 16\sqrt{11} - 53) \\ & (19\sqrt{11} - 63, -3\sqrt{11} + 10) \end{aligned}$$

...

$$\begin{array}{l} A \downarrow \\ A \downarrow \\ B \downarrow \end{array} \begin{array}{l} (1, \sqrt{11}) \\ (1, \sqrt{11} - 1) \\ (1, \sqrt{11} - 2) \\ (\sqrt{11} - 3, 1) \\ (\sqrt{11} - 3, -\sqrt{11} + 4) \\ (\sqrt{11} - 3, -2\sqrt{11} + 7) \\ (-3\sqrt{11} + 10, \sqrt{11} - 3) \\ (-3\sqrt{11} + 10, 4\sqrt{11} - 13) \\ (-3\sqrt{11} + 10, 7\sqrt{11} - 23) \\ (-3\sqrt{11} + 10, 10\sqrt{11} - 33) \\ (-3\sqrt{11} + 10, 13\sqrt{11} - 43) \\ (-3\sqrt{11} + 10, 16\sqrt{11} - 53) \\ (19\sqrt{11} - 63, -3\sqrt{11} + 10) \end{array}$$

...

$$\begin{array}{l}
A \downarrow (1, \sqrt{11}) \\
A \downarrow (1, \sqrt{11} - 1) \\
B \downarrow (1, \sqrt{11} - 2) \\
A \downarrow (\sqrt{11} - 3, 1) \\
(\sqrt{11} - 3, -\sqrt{11} + 4) \\
(\sqrt{11} - 3, -2\sqrt{11} + 7) \\
(-3\sqrt{11} + 10, \sqrt{11} - 3) \\
(-3\sqrt{11} + 10, 4\sqrt{11} - 13) \\
(-3\sqrt{11} + 10, 7\sqrt{11} - 23) \\
(-3\sqrt{11} + 10, 10\sqrt{11} - 33) \\
(-3\sqrt{11} + 10, 13\sqrt{11} - 43) \\
(-3\sqrt{11} + 10, 16\sqrt{11} - 53) \\
(19\sqrt{11} - 63, -3\sqrt{11} + 10)
\end{array}$$

...

$$\begin{array}{l}
A \downarrow (1, \sqrt{11}) \\
A \downarrow (1, \sqrt{11} - 1) \\
B \downarrow (1, \sqrt{11} - 2) \\
A \downarrow (\sqrt{11} - 3, 1) \\
A \downarrow (\sqrt{11} - 3, -\sqrt{11} + 4) \\
A \downarrow (\sqrt{11} - 3, -2\sqrt{11} + 7) \\
(-3\sqrt{11} + 10, \sqrt{11} - 3) \\
(-3\sqrt{11} + 10, 4\sqrt{11} - 13) \\
(-3\sqrt{11} + 10, 7\sqrt{11} - 23) \\
(-3\sqrt{11} + 10, 10\sqrt{11} - 33) \\
(-3\sqrt{11} + 10, 13\sqrt{11} - 43) \\
(-3\sqrt{11} + 10, 16\sqrt{11} - 53) \\
(19\sqrt{11} - 63, -3\sqrt{11} + 10)
\end{array}$$

...

$$\begin{array}{l}
A \downarrow (1, \sqrt{11}) \\
A \downarrow (1, \sqrt{11} - 1) \\
B \downarrow (1, \sqrt{11} - 2) \\
A \downarrow (\sqrt{11} - 3, 1) \\
A \downarrow (\sqrt{11} - 3, -\sqrt{11} + 4) \\
A \downarrow (\sqrt{11} - 3, -2\sqrt{11} + 7) \\
B \downarrow (-3\sqrt{11} + 10, \sqrt{11} - 3) \\
(-3\sqrt{11} + 10, 4\sqrt{11} - 13) \\
(-3\sqrt{11} + 10, 7\sqrt{11} - 23) \\
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...

$$\begin{array}{l}
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A \downarrow (\sqrt{11} - 3, -\sqrt{11} + 4) \\
A \downarrow (\sqrt{11} - 3, -2\sqrt{11} + 7) \\
B \downarrow (-3\sqrt{11} + 10, \sqrt{11} - 3) \\
A \downarrow (-3\sqrt{11} + 10, 4\sqrt{11} - 13) \\
(-3\sqrt{11} + 10, 7\sqrt{11} - 23) \\
(-3\sqrt{11} + 10, 10\sqrt{11} - 33) \\
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\end{array}$$

...

$$\begin{array}{l}
A \downarrow (1, \sqrt{11}) \\
A \downarrow (1, \sqrt{11} - 1) \\
B \downarrow (1, \sqrt{11} - 2) \\
A \downarrow (\sqrt{11} - 3, 1) \\
A \downarrow (\sqrt{11} - 3, -\sqrt{11} + 4) \\
A \downarrow (\sqrt{11} - 3, -2\sqrt{11} + 7) \\
B \downarrow (-3\sqrt{11} + 10, \sqrt{11} - 3) \\
A \downarrow (-3\sqrt{11} + 10, 4\sqrt{11} - 13) \\
A \downarrow (-3\sqrt{11} + 10, 7\sqrt{11} - 23) \\
(-3\sqrt{11} + 10, 10\sqrt{11} - 33) \\
(-3\sqrt{11} + 10, 13\sqrt{11} - 43) \\
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\end{array}$$

...



$$\begin{array}{l}
A \downarrow (1, \sqrt{11}) \\
A \downarrow (1, \sqrt{11} - 1) \\
B \downarrow (1, \sqrt{11} - 2) \\
A \downarrow (\sqrt{11} - 3, 1) \\
A \downarrow (\sqrt{11} - 3, -\sqrt{11} + 4) \\
A \downarrow (\sqrt{11} - 3, -2\sqrt{11} + 7) \\
B \downarrow (-3\sqrt{11} + 10, \sqrt{11} - 3) \\
A \downarrow (-3\sqrt{11} + 10, 4\sqrt{11} - 13) \\
A \downarrow (-3\sqrt{11} + 10, 7\sqrt{11} - 23) \\
A \downarrow (-3\sqrt{11} + 10, 10\sqrt{11} - 33) \\
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A \downarrow (-3\sqrt{11} + 10, 16\sqrt{11} - 53) \\
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\end{array}$$

...

$$\begin{array}{l}
A \downarrow (1, \sqrt{11}) \\
A \downarrow (1, \sqrt{11} - 1) \\
B \downarrow (1, \sqrt{11} - 2) \\
A \downarrow (\sqrt{11} - 3, 1) \\
A \downarrow (\sqrt{11} - 3, -\sqrt{11} + 4) \\
A \downarrow (\sqrt{11} - 3, -2\sqrt{11} + 7) \\
B \downarrow (-3\sqrt{11} + 10, \sqrt{11} - 3) \\
A \downarrow (-3\sqrt{11} + 10, 4\sqrt{11} - 13) \\
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A \downarrow (-3\sqrt{11} + 10, 13\sqrt{11} - 43) \\
A \downarrow (-3\sqrt{11} + 10, 16\sqrt{11} - 53) \\
B \downarrow (19\sqrt{11} - 63, -3\sqrt{11} + 10)
\end{array}$$

...

**Développement de  $(1, \sqrt{11})$  :**

$A, A, B, A, A, B, A, A, A, A, A, B, \dots$

$(1, \frac{42}{15}) : ABBAAABAAAAAAAAAAAAAAAAAAAAA...$

$(1, \sqrt{3}) : BBABBABBABBABBABBABBABB...$

$(1, \sqrt{5}) : ABAAABAAABAAABAAABAAABAAABA...$

$(1, \sqrt{11}) : AABAABAAAAABAABAAAAABAABAA...$

$(1, e) : ABBABBBAAABBBAAAAABBBAAAAA...$

$(1, \frac{\pi}{e}) : BAAAAABABABBABAAAAABAAAAA...$

$(1, \frac{42}{15}) : ABBAAABAAAAAAAAAAAAAAAAAAAAA...$

$(1, \sqrt{3}) : BBABBABBABBABBABBABBABB...$

$(1, \sqrt{5}) : ABAAABAAABAAABAAABAAABAAABA...$

$(1, \sqrt{11}) : AABAABAAAAABAABAAAAABAABAA...$

$(1, e) : ABBABBBAAABBBAAAAABBBAAAAA...$

$(1, \frac{\pi}{e}) : BAAAAABABABBABAAAAABAAAAA...$

Le développement de  $(1, x)$

- ▶ a une infinité de  $B$  ssi  $x$  est irrationnel ;

$(1, \frac{42}{15}) : ABBAAABAAAAAAAAAAAAAAAAAAAAA...$

$(1, \sqrt{3}) : BBABBABBABBABBABBABBABBABB...$

$(1, \sqrt{5}) : ABAAABAAABAAABAAABAAABAAABA...$

$(1, \sqrt{11}) : AABAABAAAAABAABAAAAABAABAA...$

$(1, e) : ABBABBBAAABBBAAAAABBBAAAAA...$

$(1, \frac{\pi}{e}) : BAAAAABABABBABAAAAABAAAAA...$

Le développement de  $(1, x)$

- ▶ a **une infinité de B** ssi  $x$  est **irrationnel** ;
- ▶ est **ultimement périodique** ssi  $x$  est **quadratique**.

$(1, \frac{42}{15})$  : *ABBAAABAAAAAAAAAAAAAAAAAAAAA...*

$(1, \sqrt{3})$  : *BBABBABBABBABBABBABBABB...*

$(1, \sqrt{5})$  : *ABAAABAAABAAABAAABAAABAAABA...*

$(1, \sqrt{11})$  : *AABAABAAAAABAABAAAAABAABAA...*

$(1, e)$  : *ABBABBBAAABBBAAAAABBBAAAAA...*

$(1, \frac{\pi}{e})$  : *BAAAAABABABBABAAAAABAAAAA...*

Le développement de  $(1, x)$

- ▶ a **une infinité de B** ssi  $x$  est **irrationnel** ;
- ▶ est **ultimement périodique** ssi  $x$  est **quadratique**.

**Algorithme d'Euclide** : fractions continues

**Euclide** (deux nombres) : plus grand – plus petit

**Brun** (trois nombres) : plus grand – 2ème plus grand

**Euclide** (deux nombres) : plus grand – plus petit

**Brun** (trois nombres) : plus grand – 2ème plus grand

(6, 9, 30)



**Euclide** (deux nombres) : plus grand – plus petit

**Brun** (trois nombres) : plus grand – 2ème plus grand

(6, 9, 30)

(6, 9, 21)

**Euclide** (deux nombres) : plus grand – plus petit

**Brun** (trois nombres) : plus grand – 2ème plus grand

(6, 9, 30)

(6, 9, 21)

(6, 9, 12)

**Euclide** (deux nombres) : plus grand – plus petit

**Brun** (trois nombres) : plus grand – 2ème plus grand

(6, 9, 30)

(6, 9, 21)

(6, 9, 12)

(3, 6, 9)

**Euclide** (deux nombres) : plus grand – plus petit

**Brun** (trois nombres) : plus grand – 2ème plus grand

(6, 9, 30)

(6, 9, 21)

(6, 9, 12)

(3, 6, 9)

(3, 3, 6)

(3, 3, 3)

(0, 3, 3)

(0, 0, **3**)

**Euclide** (deux nombres) : plus grand – plus petit

**Brun** (trois nombres) : plus grand – 2ème plus grand

(6, 9, 30)      (8, 15, 27)

(6, 9, 21)

(6, 9, 12)

(3, 6, 9)

(3, 3, 6)

(3, 3, 3)

(0, 3, 3)

(0, 0, **3**)

**Euclide** (deux nombres) : plus grand – plus petit

**Brun** (trois nombres) : plus grand – 2ème plus grand

(6, 9, 30)      (8, 15, 27)

(6, 9, 21)      (8, 12, 15)

(6, 9, 12)      (3, 8, 12)

(3, 6, 9)      (3, 4, 8)

(3, 3, 6)      (3, 4, 4)

(3, 3, 3)      (0, 3, 4)

(0, 3, 3)      (0, 1, 3)

(0, 0, **3**)      (0, 1, 2)

(0, 1, 1)

(0, 0, **1**)

**Euclide** (deux nombres) : plus grand – plus petit

**Brun** (trois nombres) : plus grand – 2ème plus grand

(6, 9, 30)      (8, 15, 27)

(6, 9, 21)      (8, 12, 15)

(6, 9, 12)      (3, 8, 12)

(3, 6, 9)      (3, 4, 8)

(3, 3, 6)      (3, 4, 4)

(3, 3, 3)      (0, 3, 4)

(0, 3, 3)      (0, 1, 3)

(0, 0, **3**)      (0, 1, 2)

(0, 1, 1)

(0, 0, **1**)

**Algorithme de Brun** : PGCD( $x, y, z$ )

$$(1, \sqrt{3}, \pi^2)$$



$$\begin{aligned}
& (1, \sqrt{3}, \pi^2) \\
& (1, \sqrt{3}, \pi^2 - \sqrt{3}) \\
& (1, \sqrt{3}, \pi^2 - 2\sqrt{3}) \\
& (1, \sqrt{3}, \pi^2 - 3\sqrt{3}) \\
& (1, \sqrt{3}, \pi^2 - 4\sqrt{3}) \\
& (1, \pi^2 - 5\sqrt{3}, \sqrt{3}) \\
& (-\pi^2 + 6\sqrt{3}, 1, \pi^2 - 5\sqrt{3}) \\
& (\pi^2 - 5\sqrt{3} - 1, -\pi^2 + 6\sqrt{3}, 1) \\
& (\pi^2 - 5\sqrt{3} - 1, \pi^2 - 6\sqrt{3} + 1, -\pi^2 + 6\sqrt{3}) \\
& (-2\pi^2 + 12\sqrt{3} - 1, \pi^2 - 5\sqrt{3} - 1, \pi^2 - 6\sqrt{3} + 1) \\
& (-2\pi^2 + 12\sqrt{3} - 1, \pi^2 - 5\sqrt{3} - 1, -\sqrt{3} + 2) \\
& (-2\pi^2 + 12\sqrt{3} - 1, -\pi^2 + 4\sqrt{3} + 3, \pi^2 - 5\sqrt{3} - 1) \\
& (-2\pi^2 + 12\sqrt{3} - 1, -\pi^2 + 4\sqrt{3} + 3, 2\pi^2 - 9\sqrt{3} - 4) \\
& (-2\pi^2 + 12\sqrt{3} - 1, -\pi^2 + 4\sqrt{3} + 3, 3\pi^2 - 13\sqrt{3} - 7) \\
& (4\pi^2 - 17\sqrt{3} - 10, -2\pi^2 + 12\sqrt{3} - 1, -\pi^2 + 4\sqrt{3} + 3)
\end{aligned}$$

$$\begin{aligned}
 & A \downarrow (1, \sqrt{3}, \pi^2) \\
 & (1, \sqrt{3}, \pi^2 - \sqrt{3}) \\
 & (1, \sqrt{3}, \pi^2 - 2\sqrt{3}) \\
 & (1, \sqrt{3}, \pi^2 - 3\sqrt{3}) \\
 & (1, \sqrt{3}, \pi^2 - 4\sqrt{3}) \\
 & (1, \pi^2 - 5\sqrt{3}, \sqrt{3}) \\
 & (-\pi^2 + 6\sqrt{3}, 1, \pi^2 - 5\sqrt{3}) \\
 & (\pi^2 - 5\sqrt{3} - 1, -\pi^2 + 6\sqrt{3}, 1) \\
 & (\pi^2 - 5\sqrt{3} - 1, \pi^2 - 6\sqrt{3} + 1, -\pi^2 + 6\sqrt{3}) \\
 & (-2\pi^2 + 12\sqrt{3} - 1, \pi^2 - 5\sqrt{3} - 1, \pi^2 - 6\sqrt{3} + 1) \\
 & (-2\pi^2 + 12\sqrt{3} - 1, \pi^2 - 5\sqrt{3} - 1, -\sqrt{3} + 2) \\
 & (-2\pi^2 + 12\sqrt{3} - 1, -\pi^2 + 4\sqrt{3} + 3, \pi^2 - 5\sqrt{3} - 1) \\
 & (-2\pi^2 + 12\sqrt{3} - 1, -\pi^2 + 4\sqrt{3} + 3, 2\pi^2 - 9\sqrt{3} - 4) \\
 & (-2\pi^2 + 12\sqrt{3} - 1, -\pi^2 + 4\sqrt{3} + 3, 3\pi^2 - 13\sqrt{3} - 7) \\
 & (4\pi^2 - 17\sqrt{3} - 10, -2\pi^2 + 12\sqrt{3} - 1, -\pi^2 + 4\sqrt{3} + 3)
 \end{aligned}$$

$$\begin{array}{l}
A \downarrow \\
A \downarrow \\
(1, \sqrt{3}, \pi^2) \\
(1, \sqrt{3}, \pi^2 - \sqrt{3}) \\
(1, \sqrt{3}, \pi^2 - 2\sqrt{3}) \\
(1, \sqrt{3}, \pi^2 - 3\sqrt{3}) \\
(1, \sqrt{3}, \pi^2 - 4\sqrt{3}) \\
(1, \pi^2 - 5\sqrt{3}, \sqrt{3}) \\
(-\pi^2 + 6\sqrt{3}, 1, \pi^2 - 5\sqrt{3}) \\
(\pi^2 - 5\sqrt{3} - 1, -\pi^2 + 6\sqrt{3}, 1) \\
(\pi^2 - 5\sqrt{3} - 1, \pi^2 - 6\sqrt{3} + 1, -\pi^2 + 6\sqrt{3}) \\
(-2\pi^2 + 12\sqrt{3} - 1, \pi^2 - 5\sqrt{3} - 1, \pi^2 - 6\sqrt{3} + 1) \\
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\end{array}$$

**Brun-développement de  $(1, \sqrt{3}, \pi^2)$  :**

*AAAABCCBCBAAACCCACCBBBC...*

$(1, 8, 26) : AABAAACABAAAAAAAAAAAAAAAAAAAAA\dots$

$(1, 2, \sqrt{5}) : CACAAABAAABAAABAAABAAAB\dots$

$(1, 2, e) : CACBABBBAABBBAAAAABBBAAAA\dots$

$(1, 8, 26) : AABAAACABAAAAAAAAAAAAAAAAAAAAA \dots$

$(1, 2, \sqrt{5}) : CACAAABAAABAAABAAABAAAB \dots$

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$(1, \sqrt[3]{2}, \sqrt[3]{4}) : C \overline{CAACBAABCACBCCCBACAACBAAB}$

$(1, \sqrt[3]{2}, \sqrt[3]{5}) : CBBBACCBCBAAAAACACAAACCCB \dots$

$(1, \sqrt{3}, \pi^2) : AAAABCCBCABAACCCACCBBBC \dots$

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Le développement de  $(1, x, y)$

- ▶ a une infinité de  $B$  ssi  $\dim_{\mathbb{Q}}(1, x, y) \geq 2$  ;

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**Brun** : fractions continues « multidimensionnelles »

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**Brun** : fractions continues « multidimensionnelles »

Questions :

- ▶ Joli théorème de **périodicité** ? (nombres **cubiques**)

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$(1, \sqrt[3]{2}, \sqrt[3]{5}) : CBBBACCBCBAAAAAACACAAACCCCB\dots$

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**Brun** : fractions continues « multidimensionnelles »

Questions :

- ▶ Joli théorème de **périodicité** ? (nombres **cubiques**)
- ▶ **Non-canonicité** de Brun
  - ➔ mêmes questions pour les autres algos



# Références

- ▶ **Euclide**, Éléments, livre VII
- ▶ **Viggo Brun**, Algorithmes euclidiens pour trois et quatre nombres
- ▶ **Fritz Schweiger**, Multidimensional Continued Fractions
- ▶ **Valérie Berthé**, Multidimensional Euclidean algorithms, numeration and substitutions
- ▶ **V. Berthé, V. Delecroix**, S-adic expansions: a combinatorial, arithmetic and geometric approach