Journées du GDR IM 2014

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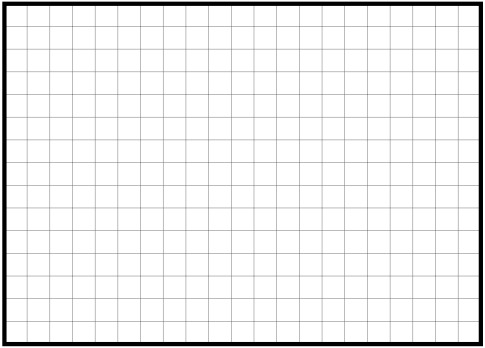
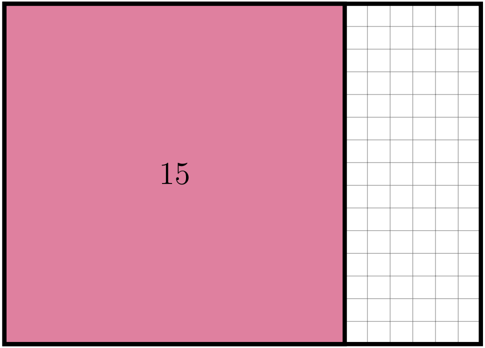
2014-01-29

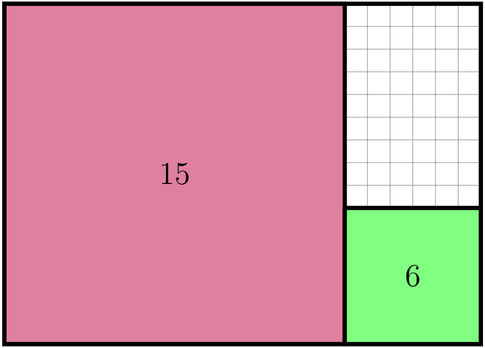


$(15, 21)$

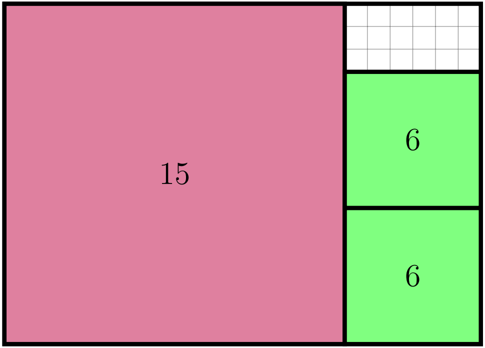


$(15, 21)$

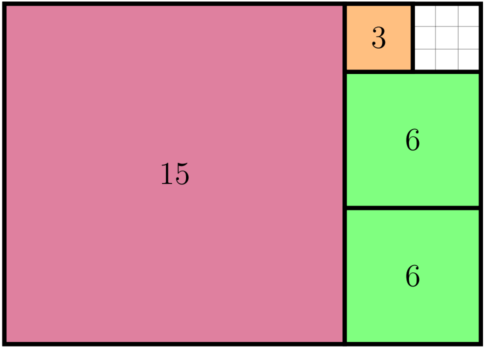
$(6, 15)$



$(15, 21)$
$(6, 15)$
$(6, 9)$



$(15, 21)$
$(6, 15)$
$(6, 9)$
$(3, 6)$



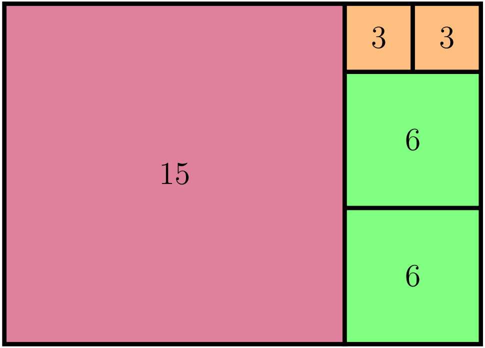
$(15, 21)$

$(6, 15)$

$(6, 9)$

$(3, 6)$

$(3, 3)$



$(15, 21)$

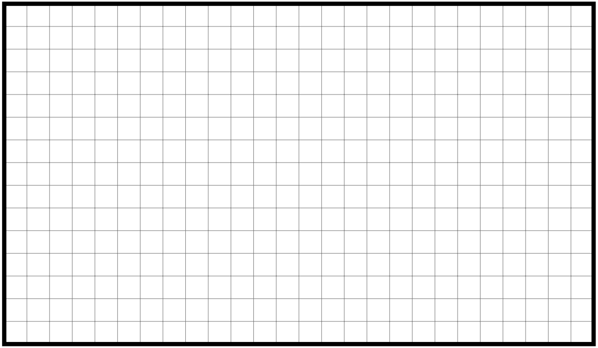
$(6, 15)$

$(6, 9)$

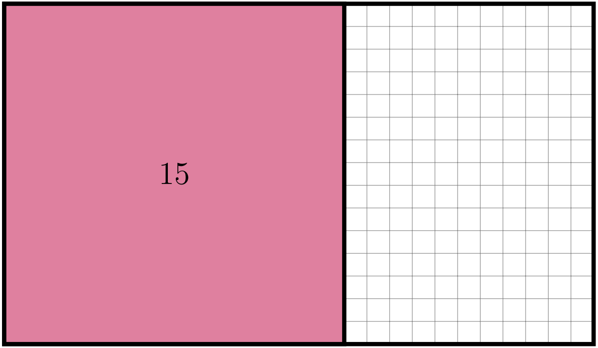
$(3, 6)$

$(3, 3)$

$(0, \mathbf{3})$

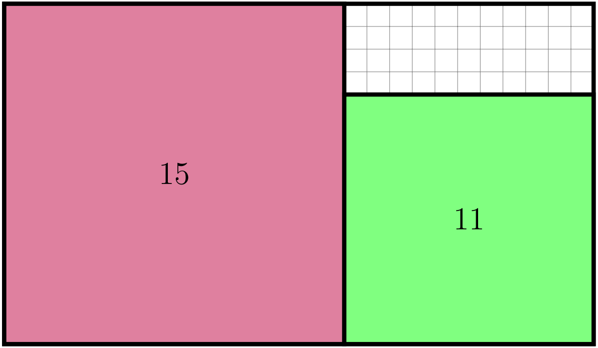


$(15, 26)$



$(15, 26)$

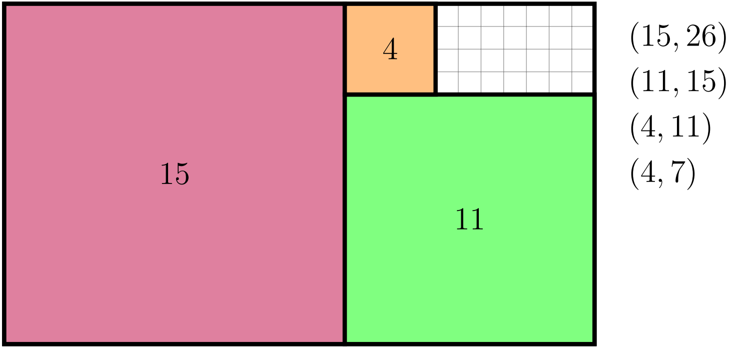
$(11, 15)$

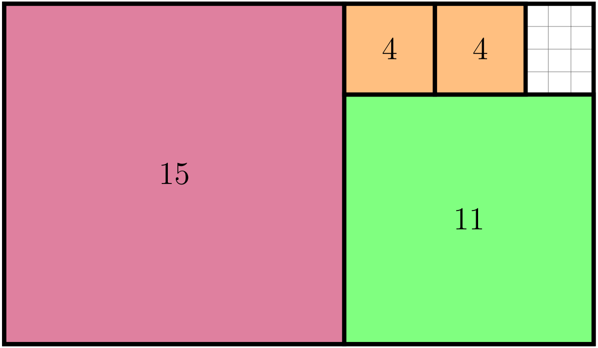


$(15, 26)$

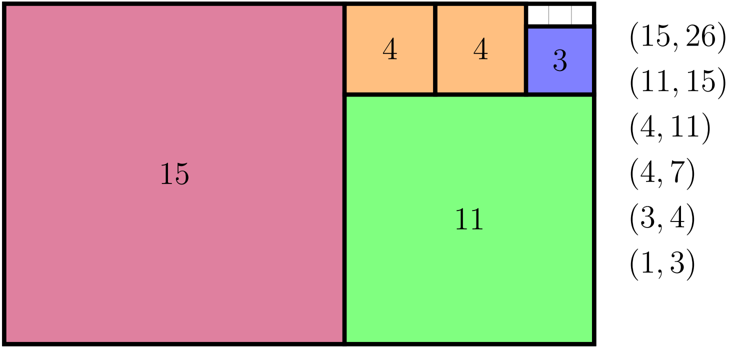
$(11, 15)$

$(4, 11)$

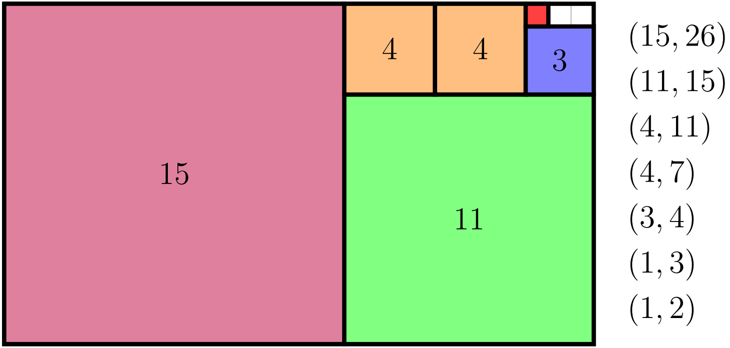

4
15
11
(15, 26)
(11, 15)
(4, 11)
(4, 7)

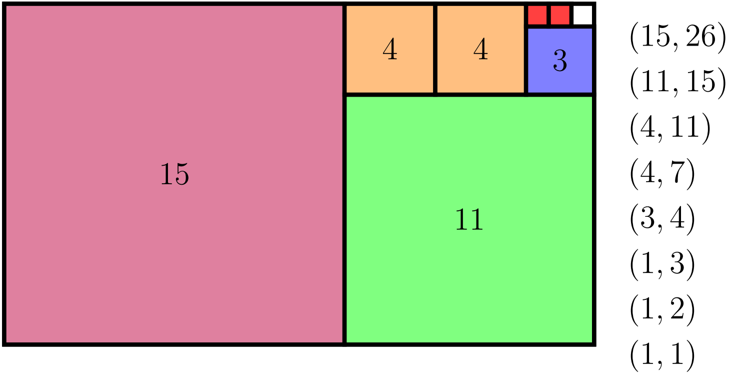


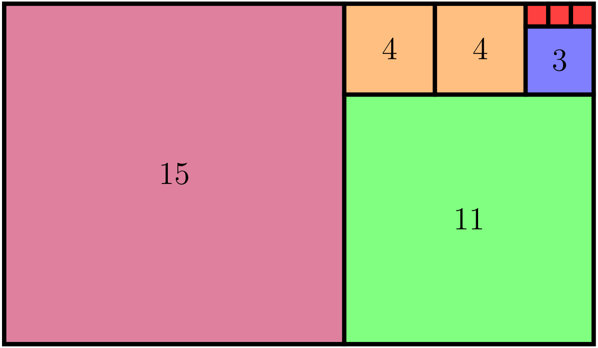
$(15, 26)$
$(11, 15)$
$(4, 11)$
$(4, 7)$
$(3, 4)$



15
4
4
3
11
(15, 26)
(11, 15)
(4, 11)
(4, 7)
(3, 4)
(1, 3)


15
4
4
3
11
(15, 26)
(11, 15)
(4, 11)
(4, 7)
(3, 4)
(1, 3)
(1, 2)


15
4
4
3
11
(15, 26)
(11, 15)
(4, 11)
(4, 7)
(3, 4)
(1, 3)
(1, 2)
(1, 1)



$(15, 26)$

$(11, 15)$

$(4, 11)$

$(4, 7)$

$(3, 4)$

$(1, 3)$

$(1, 2)$

$(1, 1)$

$(0, \mathbf{1})$

Algorithme d'Euclide : $\mathsf{PGCD}(x, y)$

$(1,\ \sqrt{11})$

$$(1,\ \sqrt{11})$$
$$(1,\ \sqrt{11}-1)$$

$$(1,\ \sqrt{11})$$
$$(1,\ \sqrt{11}-1)$$
$$(1,\ \sqrt{11}-2)$$

$$(1,\ \sqrt{11})$$
$$(1,\ \sqrt{11}-1)$$
$$(1,\ \sqrt{11}-2)$$
$$(\sqrt{11}-3,\ 1)$$

$$(1,\ \sqrt{11})$$
$$(1,\ \sqrt{11}-1)$$
$$(1,\ \sqrt{11}-2)$$
$$(\sqrt{11}-3,\ 1)$$
$$(\sqrt{11}-3,\ -\sqrt{11}+4)$$

$$(1,\ \sqrt{11})$$
$$(1,\ \sqrt{11}-1)$$
$$(1,\ \sqrt{11}-2)$$
$$(\sqrt{11}-3,\ 1)$$
$$(\sqrt{11}-3,\ -\sqrt{11}+4)$$
$$(\sqrt{11}-3,\ -2\sqrt{11}+7)$$

$$(1,\ \sqrt{11})$$
$$(1,\ \sqrt{11}-1)$$
$$(1,\ \sqrt{11}-2)$$
$$(\sqrt{11}-3,\ 1)$$
$$(\sqrt{11}-3,\ -\sqrt{11}+4)$$
$$(\sqrt{11}-3,\ -2\sqrt{11}+7)$$
$$(-3\sqrt{11}+10,\ \sqrt{11}-3)$$

$$
\begin{aligned}
&(1,\ \sqrt{11})\\
&(1,\ \sqrt{11}-1)\\
&(1,\ \sqrt{11}-2)\\
&(\sqrt{11}-3,\ 1)\\
&(\sqrt{11}-3,\ -\sqrt{11}+4)\\
&(\sqrt{11}-3,\ -2\sqrt{11}+7)\\
&(-3\sqrt{11}+10,\ \sqrt{11}-3)\\
&(-3\sqrt{11}+10,\ 4\sqrt{11}-13)\\
&(-3\sqrt{11}+10,\ 7\sqrt{11}-23)\\
&(-3\sqrt{11}+10,\ 10\sqrt{11}-33)\\
&(-3\sqrt{11}+10,\ 13\sqrt{11}-43)\\
&(-3\sqrt{11}+10,\ 16\sqrt{11}-53)\\
&(19\sqrt{11}-63,\ -3\sqrt{11}+10)\\
&\ldots
\end{aligned}
$$

$$(1,\ \sqrt{11})$$

A

$$(1,\ \sqrt{11}-1)$$
$$(1,\ \sqrt{11}-2)$$
$$(\sqrt{11}-3,\ 1)$$
$$(\sqrt{11}-3,\ -\sqrt{11}+4)$$
$$(\sqrt{11}-3,\ -2\sqrt{11}+7)$$
$$(-3\sqrt{11}+10,\ \sqrt{11}-3)$$
$$(-3\sqrt{11}+10,\ 4\sqrt{11}-13)$$
$$(-3\sqrt{11}+10,\ 7\sqrt{11}-23)$$
$$(-3\sqrt{11}+10,\ 10\sqrt{11}-33)$$
$$(-3\sqrt{11}+10,\ 13\sqrt{11}-43)$$
$$(-3\sqrt{11}+10,\ 16\sqrt{11}-53)$$
$$(19\sqrt{11}-63,\ -3\sqrt{11}+10)$$
$$\ldots$$

$$
\begin{array}{rl}
 & (1,\ \sqrt{11}) \\
A \downarrow & (1,\ \sqrt{11}-1) \\
A \downarrow & (1,\ \sqrt{11}-2) \\
 & (\sqrt{11}-3,\ 1) \\
 & (\sqrt{11}-3,\ -\sqrt{11}+4) \\
 & (\sqrt{11}-3,\ -2\sqrt{11}+7) \\
 & (-3\sqrt{11}+10,\ \sqrt{11}-3) \\
 & (-3\sqrt{11}+10,\ 4\sqrt{11}-13) \\
 & (-3\sqrt{11}+10,\ 7\sqrt{11}-23) \\
 & (-3\sqrt{11}+10,\ 10\sqrt{11}-33) \\
 & (-3\sqrt{11}+10,\ 13\sqrt{11}-43) \\
 & (-3\sqrt{11}+10,\ 16\sqrt{11}-53) \\
 & (19\sqrt{11}-63,\ -3\sqrt{11}+10) \\
 & \ldots
\end{array}
$$

$$
\begin{array}{rl}
 & (1,\ \sqrt{11}) \\
A \downarrow & (1,\ \sqrt{11}-1) \\
A \downarrow & (1,\ \sqrt{11}-2) \\
B \downarrow & (\sqrt{11}-3,\ 1) \\
 & (\sqrt{11}-3,\ -\sqrt{11}+4) \\
 & (\sqrt{11}-3,\ -2\sqrt{11}+7) \\
 & (-3\sqrt{11}+10,\ \sqrt{11}-3) \\
 & (-3\sqrt{11}+10,\ 4\sqrt{11}-13) \\
 & (-3\sqrt{11}+10,\ 7\sqrt{11}-23) \\
 & (-3\sqrt{11}+10,\ 10\sqrt{11}-33) \\
 & (-3\sqrt{11}+10,\ 13\sqrt{11}-43) \\
 & (-3\sqrt{11}+10,\ 16\sqrt{11}-53) \\
 & (19\sqrt{11}-63,\ -3\sqrt{11}+10) \\
 & \ldots
\end{array}
$$

$$
\begin{aligned}
& (1,\ \sqrt{11}) \\
A \downarrow\ & (1,\ \sqrt{11}-1) \\
A \downarrow\ & (1,\ \sqrt{11}-2) \\
B \downarrow\ & (\sqrt{11}-3,\ 1) \\
A \downarrow\ & (\sqrt{11}-3,\ -\sqrt{11}+4) \\
& (\sqrt{11}-3,\ -2\sqrt{11}+7) \\
& (-3\sqrt{11}+10,\ \sqrt{11}-3) \\
& (-3\sqrt{11}+10,\ 4\sqrt{11}-13) \\
& (-3\sqrt{11}+10,\ 7\sqrt{11}-23) \\
& (-3\sqrt{11}+10,\ 10\sqrt{11}-33) \\
& (-3\sqrt{11}+10,\ 13\sqrt{11}-43) \\
& (-3\sqrt{11}+10,\ 16\sqrt{11}-53) \\
& (19\sqrt{11}-63,\ -3\sqrt{11}+10) \\
& \ldots
\end{aligned}
$$

$$
\begin{array}{rl}
 & (1,\ \sqrt{11}) \\
A \downarrow & (1,\ \sqrt{11}-1) \\
A \downarrow & (1,\ \sqrt{11}-2) \\
B \downarrow & (\sqrt{11}-3,\ 1) \\
A \downarrow & (\sqrt{11}-3,\ -\sqrt{11}+4) \\
A \downarrow & (\sqrt{11}-3,\ -2\sqrt{11}+7) \\
 & (-3\sqrt{11}+10,\ \sqrt{11}-3) \\
 & (-3\sqrt{11}+10,\ 4\sqrt{11}-13) \\
 & (-3\sqrt{11}+10,\ 7\sqrt{11}-23) \\
 & (-3\sqrt{11}+10,\ 10\sqrt{11}-33) \\
 & (-3\sqrt{11}+10,\ 13\sqrt{11}-43) \\
 & (-3\sqrt{11}+10,\ 16\sqrt{11}-53) \\
 & (19\sqrt{11}-63,\ -3\sqrt{11}+10) \\
 & \ldots
\end{array}
$$

$$
\begin{array}{rl}
 & (1,\ \sqrt{11}) \\
A \downarrow & \\
 & (1,\ \sqrt{11}-1) \\
A \downarrow & \\
 & (1,\ \sqrt{11}-2) \\
B \downarrow & \\
 & (\sqrt{11}-3,\ 1) \\
A \downarrow & \\
 & (\sqrt{11}-3,\ -\sqrt{11}+4) \\
A \downarrow & \\
 & (\sqrt{11}-3,\ -2\sqrt{11}+7) \\
B \downarrow & \\
 & (-3\sqrt{11}+10,\ \sqrt{11}-3) \\
 & (-3\sqrt{11}+10,\ 4\sqrt{11}-13) \\
 & (-3\sqrt{11}+10,\ 7\sqrt{11}-23) \\
 & (-3\sqrt{11}+10,\ 10\sqrt{11}-33) \\
 & (-3\sqrt{11}+10,\ 13\sqrt{11}-43) \\
 & (-3\sqrt{11}+10,\ 16\sqrt{11}-53) \\
 & (19\sqrt{11}-63,\ -3\sqrt{11}+10) \\
 & \ldots
\end{array}
$$

$$
\begin{array}{rl}
 & (1,\ \sqrt{11}) \\
A \downarrow & (1,\ \sqrt{11}-1) \\
A \downarrow & (1,\ \sqrt{11}-2) \\
B \downarrow & (\sqrt{11}-3,\ 1) \\
A \downarrow & (\sqrt{11}-3,\ -\sqrt{11}+4) \\
A \downarrow & (\sqrt{11}-3,\ -2\sqrt{11}+7) \\
B \downarrow & (-3\sqrt{11}+10,\ \sqrt{11}-3) \\
A \downarrow & (-3\sqrt{11}+10,\ 4\sqrt{11}-13) \\
 & (-3\sqrt{11}+10,\ 7\sqrt{11}-23) \\
 & (-3\sqrt{11}+10,\ 10\sqrt{11}-33) \\
 & (-3\sqrt{11}+10,\ 13\sqrt{11}-43) \\
 & (-3\sqrt{11}+10,\ 16\sqrt{11}-53) \\
 & (19\sqrt{11}-63,\ -3\sqrt{11}+10) \\
 & \ldots
\end{array}
$$

$$
\begin{array}{ll}
 & (1,\ \sqrt{11}) \\
A \downarrow & \\
 & (1,\ \sqrt{11}-1) \\
A \downarrow & \\
 & (1,\ \sqrt{11}-2) \\
B \downarrow & \\
 & (\sqrt{11}-3,\ 1) \\
A \downarrow & \\
 & (\sqrt{11}-3,\ -\sqrt{11}+4) \\
A \downarrow & \\
 & (\sqrt{11}-3,\ -2\sqrt{11}+7) \\
B \downarrow & \\
 & (-3\sqrt{11}+10,\ \sqrt{11}-3) \\
A \downarrow & \\
 & (-3\sqrt{11}+10,\ 4\sqrt{11}-13) \\
A \downarrow & \\
 & (-3\sqrt{11}+10,\ 7\sqrt{11}-23) \\
 & (-3\sqrt{11}+10,\ 10\sqrt{11}-33) \\
 & (-3\sqrt{11}+10,\ 13\sqrt{11}-43) \\
 & (-3\sqrt{11}+10,\ 16\sqrt{11}-53) \\
 & (19\sqrt{11}-63,\ -3\sqrt{11}+10) \\
 & \ldots
\end{array}
$$

$$
\begin{array}{rl}
 & (1,\ \sqrt{11}) \\
A \downarrow & (1,\ \sqrt{11}-1) \\
A \downarrow & (1,\ \sqrt{11}-2) \\
B \downarrow & (\sqrt{11}-3,\ 1) \\
A \downarrow & (\sqrt{11}-3,\ -\sqrt{11}+4) \\
A \downarrow & (\sqrt{11}-3,\ -2\sqrt{11}+7) \\
B \downarrow & (-3\sqrt{11}+10,\ \sqrt{11}-3) \\
A \downarrow & (-3\sqrt{11}+10,\ 4\sqrt{11}-13) \\
A \downarrow & (-3\sqrt{11}+10,\ 7\sqrt{11}-23) \\
A \downarrow & (-3\sqrt{11}+10,\ 10\sqrt{11}-33) \\
A \downarrow & (-3\sqrt{11}+10,\ 13\sqrt{11}-43) \\
A \downarrow & (-3\sqrt{11}+10,\ 16\sqrt{11}-53) \\
B \downarrow & (19\sqrt{11}-63,\ -3\sqrt{11}+10) \\
 & \ldots
\end{array}
$$

$$
\begin{array}{rl}
 & (1,\ \sqrt{11}) \\
A \downarrow & (1,\ \sqrt{11}-1) \\
A \downarrow & (1,\ \sqrt{11}-2) \\
B \downarrow & (\sqrt{11}-3,\ 1) \\
A \downarrow & (\sqrt{11}-3,\ -\sqrt{11}+4) \\
A \downarrow & (\sqrt{11}-3,\ -2\sqrt{11}+7) \\
B \downarrow & (-3\sqrt{11}+10,\ \sqrt{11}-3) \\
A \downarrow & (-3\sqrt{11}+10,\ 4\sqrt{11}-13) \\
A \downarrow & (-3\sqrt{11}+10,\ 7\sqrt{11}-23) \\
A \downarrow & (-3\sqrt{11}+10,\ 10\sqrt{11}-33) \\
A \downarrow & (-3\sqrt{11}+10,\ 13\sqrt{11}-43) \\
A \downarrow & (-3\sqrt{11}+10,\ 16\sqrt{11}-53) \\
B \downarrow & (19\sqrt{11}-63,\ -3\sqrt{11}+10) \\
 & \dots
\end{array}
$$

Développement de $(1, \sqrt{11})$ **:**

$A, A, B, A, A, B, A, A, A, A, A, B, \dots$

$$(1, \tfrac{42}{15}) : \; ABBAAABAAAAAAAAAAAAAAAAAAAA\ldots$$

$$(1, \sqrt{3}) : \; BBABBABBABBABBABBABBABB\ldots$$

$$(1, \sqrt{5}) : \; ABAAABAAABAAABAAABAAABAAABA\ldots$$

$$(1, \sqrt{11}) : \; AABAABAAAAABAABAAAAABAABAA\ldots$$

$$(1, e) : \; ABBABBBAAABBBAAAAABBBAAAAAA\ldots$$

$$(1, \tfrac{\pi}{e}) : \; BAAAAABABABBABAAAAABAAAAA\ldots$$

$(1, \frac{42}{15})$: $ABBAAABAAAAAAAAAAAAAAAAAAAA\ldots$

$(1, \sqrt{3})$: $BBABBABBABBABBABBABBABBABB\ldots$

$(1, \sqrt{5})$: $ABAAABAAABAAABAAABAAABAAABA\ldots$

$(1, \sqrt{11})$: $AABAABAAAAABAABAAAAABAABAA\ldots$

$(1, e)$: $ABBABBBAAABBBAAAAABBBAAAAAA\ldots$

$(1, \frac{\pi}{e})$: $BAAAAABABABBABAAAAABAAAAA\ldots$

Le développement de $(1, x)$

- a **une infinité de** B ssi x est **irrationnel** ;

$(1, \frac{42}{15})$: $ABBAAABAAAAAAAAAAAAAAAAAA\ldots$

$(1, \sqrt{3})$: $BBABBABBABBABBABBABBABBABB\ldots$

$(1, \sqrt{5})$: $ABAAABAAABAAABAAABAAABAAABA\ldots$

$(1, \sqrt{11})$: $AABAABAAAAABAABAAAAABAABAA\ldots$

$(1, e)$: $ABBABBBAAABBBAAAAABBBAAAAAA\ldots$

$(1, \frac{\pi}{e})$: $BAAAAABABABBABAAAAABAAAAA\ldots$

Le développement de $(1, x)$

- a **une infinité de** B ssi x est **irrationnel** ;
- est **ultimement périodique** ssi x est **quadratique**.

$(1, \frac{42}{15})$: $ABBAAABAAAAAAAAAAAAAAAAAAAA\dots$

$(1, \sqrt{3})$: $BBABBABBABBABBABBABBABBABB\dots$

$(1, \sqrt{5})$: $ABAAABAAABAAABAAABAAABAAABA\dots$

$(1, \sqrt{11})$: $AABAABAAAAABAABAAAAABAABAA\dots$

$(1, e)$: $ABBABBBAAABBBAAAAABBBAAAAAA\dots$

$(1, \frac{\pi}{e})$: $BAAAAABABABBABAAAAABAAAAA\dots$

Le développement de $(1, x)$

- a **une infinité de** B ssi x est **irrationnel** ;
- est **ultimement périodique** ssi x est **quadratique**.

Algorithme d'Euclide : fractions continues

Euclide (deux nombres) : plus grand $-$ plus petit

Brun (trois nombres) : plus grand $-$ 2ème plus grand

Euclide (deux nombres) : plus grand $-$ plus petit

Brun (trois nombres) : plus grand $-$ 2ème plus grand

$(6, 9, 30)$

Euclide (deux nombres) : plus grand $-$ plus petit

Brun (trois nombres) : plus grand $-$ 2ème plus grand

$(6, 9, 30)$
$(6, 9, 21)$

Euclide (deux nombres) : plus grand − plus petit

Brun (trois nombres) : plus grand − 2ème plus grand

$(6, 9, 30)$
$(6, 9, 21)$
$(6, 9, 12)$

Euclide (deux nombres) : plus grand − plus petit

Brun (trois nombres) : plus grand − 2ème plus grand

$(6, 9, 30)$
$(6, 9, 21)$
$(6, 9, 12)$
$(3, 6, 9)$

Euclide (deux nombres) : plus grand − plus petit

Brun (trois nombres) : plus grand − 2ème plus grand

$(6, 9, 30)$
$(6, 9, 21)$
$(6, 9, 12)$
$(3, 6, 9)$
$(3, 3, 6)$
$(3, 3, 3)$
$(0, 3, 3)$
$(0, 0, \mathbf{3})$

Euclide (deux nombres) : plus grand $-$ plus petit

Brun (trois nombres) : plus grand $-$ 2ème plus grand

$(6,9,30)$	$(8,15,27)$
$(6,9,21)$	
$(6,9,12)$	
$(3,6,9)$	
$(3,3,6)$	
$(3,3,3)$	
$(0,3,3)$	
$(0,0,\mathbf{3})$	

Euclide (deux nombres) : plus grand − plus petit

Brun (trois nombres) : plus grand − 2ème plus grand

$(6,9,30)$	$(8,15,27)$
$(6,9,21)$	$(8,12,15)$
$(6,9,12)$	$(3,8,12)$
$(3,6,9)$	$(3,4,8)$
$(3,3,6)$	$(3,4,4)$
$(3,3,3)$	$(0,3,4)$
$(0,3,3)$	$(0,1,3)$
$(0,0,\mathbf{3})$	$(0,1,2)$
	$(0,1,1)$
	$(0,0,\mathbf{1})$

Euclide (deux nombres) : plus grand − plus petit

Brun (trois nombres) : plus grand − 2ème plus grand

$(6,9,30)$	$(8,15,27)$
$(6,9,21)$	$(8,12,15)$
$(6,9,12)$	$(3,8,12)$
$(3,6,9)$	$(3,4,8)$
$(3,3,6)$	$(3,4,4)$
$(3,3,3)$	$(0,3,4)$
$(0,3,3)$	$(0,1,3)$
$(0,0,\mathbf{3})$	$(0,1,2)$
	$(0,1,1)$
	$(0,0,\mathbf{1})$

Algorithme de Brun : PGCD(x,y,z)

$$(1,\ \sqrt{3},\ \pi^2)$$

$$(1,\ \sqrt{3},\ \pi^2)$$
$$(1,\ \sqrt{3},\ \pi^2 - \sqrt{3})$$
$$(1,\ \sqrt{3},\ \pi^2 - 2\sqrt{3})$$
$$(1,\ \sqrt{3},\ \pi^2 - 3\sqrt{3})$$
$$(1,\ \sqrt{3},\ \pi^2 - 4\sqrt{3})$$
$$(1,\ \pi^2 - 5\sqrt{3},\ \sqrt{3})$$
$$(-\pi^2 + 6\sqrt{3},\ 1,\ \pi^2 - 5\sqrt{3})$$
$$(\pi^2 - 5\sqrt{3} - 1,\ -\pi^2 + 6\sqrt{3},\ 1)$$
$$(\pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1,\ -\pi^2 + 6\sqrt{3})$$
$$(-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1)$$
$$(-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ -\sqrt{3} + 2)$$
$$(-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ \pi^2 - 5\sqrt{3} - 1)$$
$$(-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 2\pi^2 - 9\sqrt{3} - 4)$$
$$(-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 3\pi^2 - 13\sqrt{3} - 7)$$
$$(4\pi^2 - 17\sqrt{3} - 10,\ -2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3)$$

A

$$
\begin{aligned}
&(1,\ \sqrt{3},\ \pi^2)\\
&(1,\ \sqrt{3},\ \pi^2-\sqrt{3})\\
&(1,\ \sqrt{3},\ \pi^2-2\sqrt{3})\\
&(1,\ \sqrt{3},\ \pi^2-3\sqrt{3})\\
&(1,\ \sqrt{3},\ \pi^2-4\sqrt{3})\\
&(1,\ \pi^2-5\sqrt{3},\ \sqrt{3})\\
&(-\pi^2+6\sqrt{3},\ 1,\ \pi^2-5\sqrt{3})\\
&(\pi^2-5\sqrt{3}-1,\ -\pi^2+6\sqrt{3},\ 1)\\
&(\pi^2-5\sqrt{3}-1,\ \pi^2-6\sqrt{3}+1,\ -\pi^2+6\sqrt{3})\\
&(-2\pi^2+12\sqrt{3}-1,\ \pi^2-5\sqrt{3}-1,\ \pi^2-6\sqrt{3}+1)\\
&(-2\pi^2+12\sqrt{3}-1,\ \pi^2-5\sqrt{3}-1,\ -\sqrt{3}+2)\\
&(-2\pi^2+12\sqrt{3}-1,\ -\pi^2+4\sqrt{3}+3,\ \pi^2-5\sqrt{3}-1)\\
&(-2\pi^2+12\sqrt{3}-1,\ -\pi^2+4\sqrt{3}+3,\ 2\pi^2-9\sqrt{3}-4)\\
&(-2\pi^2+12\sqrt{3}-1,\ -\pi^2+4\sqrt{3}+3,\ 3\pi^2-13\sqrt{3}-7)\\
&(4\pi^2-17\sqrt{3}-10,\ -2\pi^2+12\sqrt{3}-1,\ -\pi^2+4\sqrt{3}+3)
\end{aligned}
$$

$$
\begin{array}{rl}
& (1,\ \sqrt{3},\ \pi^2) \\
A\downarrow & \\
& (1,\ \sqrt{3},\ \pi^2 - \sqrt{3}) \\
A\downarrow & \\
& (1,\ \sqrt{3},\ \pi^2 - 2\sqrt{3}) \\
& (1,\ \sqrt{3},\ \pi^2 - 3\sqrt{3}) \\
& (1,\ \sqrt{3},\ \pi^2 - 4\sqrt{3}) \\
& (1,\ \pi^2 - 5\sqrt{3},\ \sqrt{3}) \\
& (-\pi^2 + 6\sqrt{3},\ 1,\ \pi^2 - 5\sqrt{3}) \\
& (\pi^2 - 5\sqrt{3} - 1,\ -\pi^2 + 6\sqrt{3},\ 1) \\
& (\pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1,\ -\pi^2 + 6\sqrt{3}) \\
& (-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1) \\
& (-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ -\sqrt{3} + 2) \\
& (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ \pi^2 - 5\sqrt{3} - 1) \\
& (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 2\pi^2 - 9\sqrt{3} - 4) \\
& (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 3\pi^2 - 13\sqrt{3} - 7) \\
& (4\pi^2 - 17\sqrt{3} - 10,\ -2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3)
\end{array}
$$

$$\begin{array}{ll}
 & (1,\ \sqrt{3},\ \pi^2) \\
A\downarrow & \\
 & (1,\ \sqrt{3},\ \pi^2-\sqrt{3}) \\
A\downarrow & \\
 & (1,\ \sqrt{3},\ \pi^2-2\sqrt{3}) \\
A\downarrow & \\
 & (1,\ \sqrt{3},\ \pi^2-3\sqrt{3}) \\
 & (1,\ \sqrt{3},\ \pi^2-4\sqrt{3}) \\
 & (1,\ \pi^2-5\sqrt{3},\ \sqrt{3}) \\
 & (-\pi^2+6\sqrt{3},\ 1,\ \pi^2-5\sqrt{3}) \\
 & (\pi^2-5\sqrt{3}-1,\ -\pi^2+6\sqrt{3},\ 1) \\
 & (\pi^2-5\sqrt{3}-1,\ \pi^2-6\sqrt{3}+1,\ -\pi^2+6\sqrt{3}) \\
 & (-2\pi^2+12\sqrt{3}-1,\ \pi^2-5\sqrt{3}-1,\ \pi^2-6\sqrt{3}+1) \\
 & (-2\pi^2+12\sqrt{3}-1,\ \pi^2-5\sqrt{3}-1,\ -\sqrt{3}+2) \\
 & (-2\pi^2+12\sqrt{3}-1,\ -\pi^2+4\sqrt{3}+3,\ \pi^2-5\sqrt{3}-1) \\
 & (-2\pi^2+12\sqrt{3}-1,\ -\pi^2+4\sqrt{3}+3,\ 2\pi^2-9\sqrt{3}-4) \\
 & (-2\pi^2+12\sqrt{3}-1,\ -\pi^2+4\sqrt{3}+3,\ 3\pi^2-13\sqrt{3}-7) \\
 & (4\pi^2-17\sqrt{3}-10,\ -2\pi^2+12\sqrt{3}-1,\ -\pi^2+4\sqrt{3}+3)
\end{array}$$

$$\begin{array}{ll}
 & (1,\ \sqrt{3},\ \pi^2) \\
A \downarrow & (1,\ \sqrt{3},\ \pi^2 - \sqrt{3}) \\
A \downarrow & (1,\ \sqrt{3},\ \pi^2 - 2\sqrt{3}) \\
A \downarrow & (1,\ \sqrt{3},\ \pi^2 - 3\sqrt{3}) \\
A \downarrow & (1,\ \sqrt{3},\ \pi^2 - 4\sqrt{3}) \\
 & (1,\ \pi^2 - 5\sqrt{3},\ \sqrt{3}) \\
 & (-\pi^2 + 6\sqrt{3},\ 1,\ \pi^2 - 5\sqrt{3}) \\
 & (\pi^2 - 5\sqrt{3} - 1,\ -\pi^2 + 6\sqrt{3},\ 1) \\
 & (\pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1,\ -\pi^2 + 6\sqrt{3}) \\
 & (-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1) \\
 & (-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ -\sqrt{3} + 2) \\
 & (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ \pi^2 - 5\sqrt{3} - 1) \\
 & (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 2\pi^2 - 9\sqrt{3} - 4) \\
 & (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 3\pi^2 - 13\sqrt{3} - 7) \\
 & (4\pi^2 - 17\sqrt{3} - 10,\ -2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3)
\end{array}$$

$$
\begin{aligned}
&(1,\ \sqrt{3},\ \pi^2) \\
A \downarrow\ &(1,\ \sqrt{3},\ \pi^2 - \sqrt{3}) \\
A \downarrow\ &(1,\ \sqrt{3},\ \pi^2 - 2\sqrt{3}) \\
A \downarrow\ &(1,\ \sqrt{3},\ \pi^2 - 3\sqrt{3}) \\
A \downarrow\ &(1,\ \sqrt{3},\ \pi^2 - 4\sqrt{3}) \\
B \downarrow\ &(1,\ \pi^2 - 5\sqrt{3},\ \sqrt{3}) \\
&(-\pi^2 + 6\sqrt{3},\ 1,\ \pi^2 - 5\sqrt{3}) \\
&(\pi^2 - 5\sqrt{3} - 1,\ -\pi^2 + 6\sqrt{3},\ 1) \\
&(\pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1,\ -\pi^2 + 6\sqrt{3}) \\
&(-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1) \\
&(-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ -\sqrt{3} + 2) \\
&(-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ \pi^2 - 5\sqrt{3} - 1) \\
&(-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 2\pi^2 - 9\sqrt{3} - 4) \\
&(-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 3\pi^2 - 13\sqrt{3} - 7) \\
&(4\pi^2 - 17\sqrt{3} - 10,\ -2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3)
\end{aligned}
$$

$$
\begin{array}{rl}
 & (1,\ \sqrt{3},\ \pi^2) \\
A\ \big\downarrow & \\
 & (1,\ \sqrt{3},\ \pi^2 - \sqrt{3}) \\
A\ \big\downarrow & \\
 & (1,\ \sqrt{3},\ \pi^2 - 2\sqrt{3}) \\
A\ \big\downarrow & \\
 & (1,\ \sqrt{3},\ \pi^2 - 3\sqrt{3}) \\
A\ \big\downarrow & \\
 & (1,\ \sqrt{3},\ \pi^2 - 4\sqrt{3}) \\
B\ \big\downarrow & \\
 & (1,\ \pi^2 - 5\sqrt{3},\ \sqrt{3}) \\
C\ \big\downarrow & \\
 & (-\pi^2 + 6\sqrt{3},\ 1,\ \pi^2 - 5\sqrt{3}) \\
 & (\pi^2 - 5\sqrt{3} - 1,\ -\pi^2 + 6\sqrt{3},\ 1) \\
 & (\pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1,\ -\pi^2 + 6\sqrt{3}) \\
 & (-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1) \\
 & (-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ -\sqrt{3} + 2) \\
 & (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ \pi^2 - 5\sqrt{3} - 1) \\
 & (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 2\pi^2 - 9\sqrt{3} - 4) \\
 & (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 3\pi^2 - 13\sqrt{3} - 7) \\
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$$

$$
\begin{array}{rl}
 & (1,\ \sqrt{3},\ \pi^2) \\
A \downarrow & (1,\ \sqrt{3},\ \pi^2 - \sqrt{3}) \\
A \downarrow & (1,\ \sqrt{3},\ \pi^2 - 2\sqrt{3}) \\
A \downarrow & (1,\ \sqrt{3},\ \pi^2 - 3\sqrt{3}) \\
A \downarrow & (1,\ \sqrt{3},\ \pi^2 - 4\sqrt{3}) \\
B \downarrow & (1,\ \pi^2 - 5\sqrt{3},\ \sqrt{3}) \\
C \downarrow & (-\pi^2 + 6\sqrt{3},\ 1,\ \pi^2 - 5\sqrt{3}) \\
C \downarrow & (\pi^2 - 5\sqrt{3} - 1,\ -\pi^2 + 6\sqrt{3},\ 1) \\
B \downarrow & (\pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1,\ -\pi^2 + 6\sqrt{3}) \\
C \downarrow & (-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1) \\
A \downarrow & (-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ -\sqrt{3} + 2) \\
B \downarrow & (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ \pi^2 - 5\sqrt{3} - 1) \\
A \downarrow & (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 2\pi^2 - 9\sqrt{3} - 4) \\
A \downarrow & (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 3\pi^2 - 13\sqrt{3} - 7) \\
C \downarrow & (4\pi^2 - 17\sqrt{3} - 10,\ -2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3)
\end{array}
$$

$$
\begin{array}{rl}
 & (1,\ \sqrt{3},\ \pi^2) \\
A \downarrow & (1,\ \sqrt{3},\ \pi^2 - \sqrt{3}) \\
A \downarrow & (1,\ \sqrt{3},\ \pi^2 - 2\sqrt{3}) \\
A \downarrow & (1,\ \sqrt{3},\ \pi^2 - 3\sqrt{3}) \\
A \downarrow & (1,\ \sqrt{3},\ \pi^2 - 4\sqrt{3}) \\
B \downarrow & (1,\ \pi^2 - 5\sqrt{3},\ \sqrt{3}) \\
C \downarrow & (-\pi^2 + 6\sqrt{3},\ 1,\ \pi^2 - 5\sqrt{3}) \\
C \downarrow & (\pi^2 - 5\sqrt{3} - 1,\ -\pi^2 + 6\sqrt{3},\ 1) \\
B \downarrow & (\pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1,\ -\pi^2 + 6\sqrt{3}) \\
C \downarrow & (-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ \pi^2 - 6\sqrt{3} + 1) \\
A \downarrow & (-2\pi^2 + 12\sqrt{3} - 1,\ \pi^2 - 5\sqrt{3} - 1,\ -\sqrt{3} + 2) \\
B \downarrow & (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ \pi^2 - 5\sqrt{3} - 1) \\
A \downarrow & (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 2\pi^2 - 9\sqrt{3} - 4) \\
A \downarrow & (-2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3,\ 3\pi^2 - 13\sqrt{3} - 7) \\
C \downarrow & (4\pi^2 - 17\sqrt{3} - 10,\ -2\pi^2 + 12\sqrt{3} - 1,\ -\pi^2 + 4\sqrt{3} + 3)
\end{array}
$$

Brun-développement de $(1, \sqrt{3}, \pi^2)$ **:**

$AAAABCCBCABAACCCACCBBBC\ldots$

$$(1,8,26): AABAAACABAAAAAAAAAAAAAAAA\dots$$
$$(1,2,\sqrt{5}): CACAAABAAABAAABAAABAAAB\dots$$
$$(1,2,e): CACBABBBAAABBBAAAAABBBAAAA\dots$$

$$(1,8,26):\ AABAAACABAAAAAAAAAAAAAAAA\ldots$$

$$(1,2,\sqrt{5}):\ CACAAABAAABAAABAAABAAAB\ldots$$

$$(1,2,e):\ CACBABBBAAABBBAAAAABBBAAAA\ldots$$

$$(1,\sqrt[3]{2},\sqrt[3]{4}):\ C\ \overline{CAACBAABCAACBCCCCBCAACBAAB}$$

$$(1,\sqrt[3]{2},\sqrt[3]{5}):\ CBBBACCBCBAAAAAACACAAACCCB\ldots$$

$$(1,\sqrt{3},\pi^2):\ AAAABCCBCABAACCCACCBBBC\ldots$$

$(1,8,26)$: $AABAAACABAAAAAAAAAAAAAAAA\ldots$
$(1,2,\sqrt{5})$: $CACAAABAAABAAABAAABAAAB\ldots$
$(1,2,e)$: $CACBABBBAAABBBAAAAABBBAAAA\ldots$
$(1,\sqrt[3]{2},\sqrt[3]{4})$: $C\ \overline{CAACBAABCAACBCCCCBCAACBAAB}$
$(1,\sqrt[3]{2},\sqrt[3]{5})$: $CBBBACCBCBAAAAAACACAAACCCB\ldots$
$(1,\sqrt{3},\pi^2)$: $AAAABCCBCABAACCCACCBBBC\ldots$

Le développement de $(1,x,y)$

- a une infinité de B ssi $\dim_{\mathbb{Q}}(1,x,y)\geqslant 2$;

$(1,8,26)$: $AABAAACABAAAAAAAAAAAAAAAA\ldots$

$(1,2,\sqrt{5})$: $CACAAABAAABAAABAAABAAAB\ldots$

$(1,2,e)$: $CACBABBBAAABBBAAAAABBBAAAA\ldots$

$(1,\sqrt[3]{2},\sqrt[3]{4})$: $C\,\overline{CAACBAABCAACBCCCCBCAACBAAB}$

$(1,\sqrt[3]{2},\sqrt[3]{5})$: $CBBBACCBCBAAAAAACACAAACCCB\ldots$

$(1,\sqrt{3},\pi^2)$: $AAAABCCBCABAACCCACCBBBC\ldots$

Le développement de $(1,x,y)$

- a une infinité de B ssi $\dim_{\mathbb{Q}}(1,x,y) \geqslant 2$;
- a une infinité de C ssi $\dim_{\mathbb{Q}}(1,x,y) = 3$.

$(1,8,26)$: $AABAAACABAAAAAAAAAAAAAAAA\ldots$

$(1,2,\sqrt{5})$: $CACAAABAAABAAABAAABAAAB\ldots$

$(1,2,e)$: $CACBABBBAAABBBAAAAABBBAAAA\ldots$

$(1,\sqrt[3]{2},\sqrt[3]{4})$: $C\,\overline{CAACBAABCAACBCCCCBCAACBAAB}$

$(1,\sqrt[3]{2},\sqrt[3]{5})$: $CBBBACCBCBAAAAAACACAAACCCB\ldots$

$(1,\sqrt{3},\pi^2)$: $AAAABCCBCABAACCCACCBBBC\ldots$

Le développement de $(1,x,y)$

- a une infinité de B ssi $\dim_{\mathbb{Q}}(1,x,y) \geqslant 2$;
- a une infinité de C ssi $\dim_{\mathbb{Q}}(1,x,y) = 3$.

Brun : fractions continues « multidimensionnelles »

$(1,8,26)$: $AABAAACABAAAAAAAAAAAAAAAAA\ldots$
$(1,2,\sqrt{5})$: $CACAAABAAABAAABAAABAAAB\ldots$
$(1,2,e)$: $CACBABBBAAABBBAAAAABBBAAAA\ldots$
$(1,\sqrt[3]{2},\sqrt[3]{4})$: $C\ \overline{CAACBAABCAACBCCCCBCAACBAAB}$
$(1,\sqrt[3]{2},\sqrt[3]{5})$: $CBBBACCBCBAAAAAACACAAACCCB\ldots$
$(1,\sqrt{3},\pi^2)$: $AAAABCCBCABAACCCACCBBBC\ldots$

Le développement de $(1,x,y)$

- a une infinité de B ssi $\dim_{\mathbb{Q}}(1,x,y) \geqslant 2$;
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Brun : fractions continues « multidimensionnelles »

Questions :

- Joli théorème de **périodicité** ? (nombres **cubiques**)

$(1, 8, 26)$: $AABAAACABAAAAAAAAAAAAAAAAA\ldots$
$(1, 2, \sqrt{5})$: $CACAAABAAABAAABAAABAAAB\ldots$
$(1, 2, e)$: $CACBABBBAAABBBAAAAABBBAAAA\ldots$
$(1, \sqrt[3]{2}, \sqrt[3]{4})$: $C\,\overline{CAACBAABCAACBCCCCBCAACBAAB}$
$(1, \sqrt[3]{2}, \sqrt[3]{5})$: $CBBBACCBCBAAAAAACACAAACCCB\ldots$
$(1, \sqrt{3}, \pi^2)$: $AAAABCCBCABAACCCACCBBBC\ldots$

Le développement de $(1, x, y)$
- a une infinité de B ssi $\dim_{\mathbb{Q}}(1, x, y) \geqslant 2$;
- a une infinité de C ssi $\dim_{\mathbb{Q}}(1, x, y) = 3$.

Brun : fractions continues « multidimensionnelles »

Questions :
- Joli théorème de **périodicité** ? (nombres **cubiques**)
- **Non-canonicité** de Brun
  ➥ mêmes questions pour les autres algos

Références

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