Stability of limit sets of CA
a quick overview and some open problems

Timo Jolivet
Frac de printemps 2010, Caen
April, 15
Cellular automata

Definition (Cellular automaton, CA)

A function $F : A^\mathbb{Z} \rightarrow A^\mathbb{Z}$, with

- discrete time,
- discrete space $\mathbb{Z}$ and finite alphabet set $A$,
- local rule,
- same rule for every cell.
An example... 

Elementary CA 54:

001, 010, 100, 101 $\mapsto$ 1  \_ $\mapsto$ 0 

Figure: Space-time diagram of rule 54.
Limit sets

Long-term behaviour of a CA:

Definition (Limit set)

$$\Omega_F = \bigcap_{n \geq 0} F^n(A^Z)$$

...configurations that can appear arbitrarily far in time.
Limit sets

Long-term behaviour of a CA:

Definition (Limit set)

\[ \Omega_F = \bigcap_{n \geq 0} F^n(A^\mathbb{Z}) \]

...configurations that can appear arbitrarily far in time.

Examples

- \( \Omega_{\text{Id}} = A^\mathbb{Z} \)
- ECA 2: 001 \( \mapsto \) 1, 1 \( \mapsto \) 0
  \( \Omega_2 = \Sigma_{\{11,101\}} \)
- \( \Omega_{128} = \ldots \)
Limit set of rule 128

Elementary CA 128: \(111 \leftrightarrow 1 \quad \_ \leftrightarrow 0\).
Limit set of rule 128

Elementary CA 128: $111 \mapsto 1 \quad \_ \mapsto 0$.

**Proposition**

$c \in \Omega_{128}$ iff $c = \cdots 001^k 00 \cdots$ for some $k \geq 0$. 
Limit set of rule 128

Elementary CA 128: \[111 \leftrightarrow 1 \quad \_ \leftrightarrow 0.\]

**Proposition**

\[c \in \Omega_{128} \text{ iff } c = \cdots 001^k 00 \cdots \text{ for some } k \geq 0.\]

**Figure:** Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: \[ 111 \mapsto 1 \quad \_ \mapsto 0. \]

**Proposition**

\[ c \in \Omega_{128} \text{ iff } c = \cdots 001^k 00 \cdots \text{ for some } k \geq 0. \]

**Figure:** Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: $111 \mapsto 1 \quad \_ \mapsto 0$.

Proposition

$c \in \Omega_{128}$ iff $c = \cdots 001^k00\cdots$ for some $k \geq 0$.

Figure: Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: $111 \leftrightarrow 1 \quad \_ \leftrightarrow 0$.

**Proposition**

$c \in \Omega_{128}$ iff $c = \ldots 001^k00 \ldots$ for some $k \geq 0$.

**Figure:** Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: \[111 \leftrightarrow 1 \quad \_ \leftrightarrow 0.\]

**Proposition**

\[c \in \Omega_{128} \text{ iff } c = \cdots 001^k 00 \cdots \text{ for some } k \geq 0.\]

**Figure:** Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: \[111 \mapsto 1 \quad \_ \mapsto 0.\]

**Proposition**
\[c \in \Omega_{128} \text{ iff } c = \cdots 001^k 00 \cdots \text{ for some } k \geq 0.\]

*Figure:* Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: \[111 \rightarrow 1 \quad \_ \rightarrow 0.\]

Proposition
\[c \in \Omega_{128} \text{ iff } c = \cdots 001^k 00 \cdots \text{ for some } k \geq 0.\]

Figure: Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: \[111 \mapsto 1 \quad \_ \mapsto 0.\]

**Proposition**

\[c \in \Omega_{128} \text{ iff } c = \cdots 001^k 00 \cdots \text{ for some } k \geq 0.\]

**Figure:** Taking preimages by rule 128.
**Proposition**

$c \in \Omega_{128}$ iff $c = \cdots 001^k 00 \cdots$ for some $k \geq 0$.

**Figure:** Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: $111 \mapsto 1 \quad _- \mapsto 0$.

**Proposition**

$c \in \Omega_{128}$ iff $c = \cdots 001^k 00 \cdots$ for some $k \geq 0$.

**Figure:** Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: \[ 111 \mapsto 1 \quad _- \mapsto 0. \]

Proposition

\[ c \in \Omega_{128} \text{ iff } c = \cdots 001^k 00 \cdots \text{ for some } k \geq 0. \]

Figure: Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: \[ 111 \leftrightarrow 1 \quad \_ \leftrightarrow 0. \]

Proposition

\[ c \in \Omega_{128} \iff c = \cdots 001^k 00 \cdots \text{ for some } k \geq 0. \]

Figure: Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: \[111 \rightarrow 1 \quad \_ \leftrightarrow 0.\]

**Proposition**

\[c \in \Omega_{128} \text{ iff } c = \cdots 001^k 00 \cdots \text{ for some } k \geq 0.\]

**Figure:** Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: \[111 \mapsto 1 \quad \_ \mapsto 0.\]

**Proposition**

\[c \in \Omega_{128} \text{ iff } c = \cdots 001^k 00 \cdots \text{ for some } k \geq 0.\]

**Figure:** Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: \[111 \mapsto 1 \quad \_ \mapsto 0.\]

**Proposition**

\[c \in \Omega_{128} \text{ iff } c = \cdots 001^k00\cdots \text{ for some } k \geq 0.\]

**Figure:** Taking preimages by rule 128.
Limit set of rule 128

Elementary CA 128: \[ 111 \mapsto 1 \quad _- \mapsto 0. \]

Proposition

\[ c \in \Omega_{128} \text{ iff } c = \cdots 001^k 00 \cdots \text{ for some } k \geq 0. \]

Figure: Taking preimages by rule 128.
Stability

\[ A^\mathbb{Z} \supseteq F(A^\mathbb{Z}) \supseteq F^2(A^\mathbb{Z}) \supseteq \cdots \]

**Definition**

\( F \) is stable if \( \Omega_F = F^n(A^\mathbb{Z}) \) for some \( n \geq 0 \).

\( F \) is unstable otherwise.
Stability

\[ A^\mathbb{Z} \supseteq F(A^\mathbb{Z}) \supseteq F^2(A^\mathbb{Z}) \supseteq \cdots \]

**Definition**

*F* is stable if \( \Omega_F = F^n(A^\mathbb{Z}) \) for some \( n \geq 0 \).

*F* is unstable otherwise.

**Examples**

- The identity is stable.
- ECA 2 is stable: \( \Omega_2 = F(A^\mathbb{Z}) = \Sigma_{\{11,101\}} \)
- ECA 128 is not stable: for all \( k \geq 0 \).
  \( \cdots 0010^{2k}100 \cdots \) is in \( F^k(A^\mathbb{Z}) \) but not in \( \Omega_F \)
Definitions

**Nilpotent CA:** the limit set is a singleton

**Language of a subshift:** $L(S)$ is the set of words that appear in some configuration of $S$

**SFT:** subshift with a finite set of forbidden words

**Sofic subshift:** factor of SFT ($\Leftrightarrow$ regular language in 1D)

**Transitive subshift:**\[
\begin{align*}
\{ & u \in L(S), \\
& v \in L(S) \} \Rightarrow \exists w \in L(S) : uwv \in L(S)
\end{align*}
\]

**Mixing subshift:**\[
\begin{align*}
\{ & u \in L(S), \\
& v \in L(S) \} \Rightarrow \exists M \geq 0 : \forall m \geq M : \\
& \exists w \in L(S) \text{ of size } m \text{ s.t. } uwv \in L(S)
\end{align*}
\]
Nilpotent $\Rightarrow$ stable

Proposition

$F$ is nilpotent $\Rightarrow$ $F$ is stable

Proof

- Let $u$ be a "universal" configuration.
- Let $n$ such that $F^n(u) = \cdots qqq \cdots$.
- Then $F^n(A^\mathbb{Z}) = \{\cdots qqq \cdots \}$, because otherwise there exists a word whose image is not $qq \cdots q$. 
  $\blacksquare$
Proposition

$\Omega_F$ is an SFT $\implies F$ is stable

Proof

- $\Omega_F = \{c \text{ with a finite list of forbidden words } w_1, \ldots, w_n\}$
- If $F^k(A^\mathbb{Z}) = F^{k+1}(A^\mathbb{Z})$ for some $k$, then $F$ is stable.
- Otherwise, each iteration of $F$ forbids at least one new word.
  $\implies$ Contradiction because $\{w_1, \ldots, w_n\}$ is finite.
**Proposition**

\[ \Omega_F \text{ is an SFT} \implies F \text{ is stable} \]

**Proof**

- \( \Omega_F = \{ c \text{ with a finite list of forbidden words } w_1, \ldots, w_n \} \)
- If \( F^k(A^\mathbb{Z}) = F^{k+1}(A^\mathbb{Z}) \) for some \( k \), then \( F \) is stable.
- Otherwise, each iteration of \( F \) forbids at least one new word.
  - \( \Rightarrow \) Contradiction because \( \{ w_1, \ldots, w_n \} \) is finite.

...is the converse true? ...
A non-SFT limit set of a stable CA

Let $F$ be the 1DCA with state set $\{0, 1, 2\}$, neighborhood $(0, 1)$, and local rule

$$f(x, y) = \begin{cases} 
1 & \text{if } (x, y) = (0, 1) \text{ or } (0, 2) \\
2 & \text{if } (x, y) = (1, 0) \text{ or } (2, 0) \\
0 & \text{otherwise}
\end{cases}$$

Proposition

1. $F$ is stable
2. $\Omega_F$ is not of finite type

Proof

- $F(\{0, 1, 2\}^\mathbb{Z}) = \{c \text{ s.t. } 10*1 \notin c \text{ and } 20*2 \notin c\} = \Omega_F$
- left equality: $F$ forces 1 and 2 to alternate
- right equality: 1 and 2 play the same role
Necessary condition for stability

We want to characterize the limit sets of stable CA.

**Proposition**

Every limit set $S$ of a stable CA verifies:

- $S$ is sofic,
- $S$ is mixing,
- $S$ has a uniform config.
Necessary condition for stability

We want to characterize the limit sets of stable CA.

**Proposition**

Every limit set $S$ of a stable CA verifies:

- $S$ is sofic,
- $S$ is mixing,
- $S$ has a uniform config.

When are these conditions **sufficient**?
Completeness for SFTs

**Theorem [Maass]**

\[
\begin{aligned}
\text{SFT sofic, mixing, has a uniform config.} & \quad \Rightarrow \\
\text{limit set of a stable CA}
\end{aligned}
\]

**Even better:** If the SFT $S$ verifies the above hypotheses, for any factor map $G : S \to S$, we can effectively construct a stable CA $F$ such that $\Omega_F = S$ and $F|_S = G$. 
Completeness for SFTs

Theorem [Maass]

\[
\left\{ \begin{array}{c}
\text{SFT} \\
\text{sofic, mixing, has a uniform config.}
\end{array} \right\} \Rightarrow \text{limit set of a stable CA}
\]

**Even better:** If the SFT $S$ verifies the above hypotheses, for any factor map $G : S \to S$, we can effectively construct a stable CA $F$ such that $\Omega_F = S$ and $F|_S = G$.

Can we generalize this?
Completeness for AFTs

Yes, for the class of AFT subshifts [Maass].

**AFT**: natural weakening of finite type condition.

(SFT \(\subset\) AFT \(\subset\) sofic)

**Theorem [Maass]**

\[
\begin{align*}
\text{AFT} & \quad \text{sofic, mixing, has a (recept.) unif. config.} \\
\Rightarrow & \quad \text{limit set of a stable CA}
\end{align*}
\]

**Even better**: [...]

Completeness for AFTs

Yes, for the class of AFT subshifts [Maass].

**AFT:** natural weakening of finite type condition...

(SFT ⊂ AFT ⊂ sofic)

**Theorem [Maass]**

{ AFT sofic, mixing, has a (recept.) unif. config. $\Rightarrow$ limit set of a stable CA }

**Even better:** [...]

**Big Question**

Can this be extended to sofic subshifts?

$\Rightarrow$ Would be a nice characterization of stable limit sets.
Unlike in the stable case, we are far from a good understanding of unstable limit sets.
Unlike in the stable case, we are far from a good understanding of unstable limit sets.

**Big Question**

Does there exist a subshift that is the limit set of an unstable CA and of a stable CA?
Unlike in the stable case, we are far from a good understanding of unstable limit sets.

**Big Question**

Does there exist a subshift that is the limit set of an unstable CA and of a stable CA?

**The best answer so far...** [di Lena, Margara]

\[
\begin{cases}
F \text{ stable} \\
G \text{ unstable} \\
F \text{ and } G \text{ have the same limit set } \Omega
\end{cases}
\Rightarrow F|_{\Omega} \neq G|_{\Omega}.
\]
Unlike in the stable case, we are far from a good understanding of unstable limit sets.

**Big Question**

Does there exist a subshift that is the limit set of an unstable CA and of a stable CA?

The best answer so far... [di Lena, Margara]

\[
\begin{cases}
F \text{ stable} \\
G \text{ unstable} \\
F \text{ and } G \text{ have the same limit set } \Omega
\end{cases} \Rightarrow F|_\Omega \neq G|_\Omega.
\]

Some negative results...

- SFT $\Rightarrow$ cannot be unstable
- NM $\Rightarrow$ cannot be unstable [Maass]
- The even subshift $\Sigma\{01^{2k+1}10\}$ cannot be unstable
Transitivity and unstability

All the easy examples of limit sets of unstable CA are not transitive.

Does there exist a transitive limit set of an unstable CA?
Transitivity and instability

All the easy examples of limit sets of unstable CA are not transitive.

Does there exit a transitive limit set of an unstable CA?

Theorem [Formenti, Kůrka]
There exists a mixing limit set of an unstable CA.
Transitivity and unstability

All the easy examples of limit sets of unstable CA are not transitive.

Does there exit a transitive limit set of an unstable CA?

**Theorem [Formenti, Kůrka]**

There exists a mixing limit set of an unstable CA.

This example is non-sofic. Does there exist a sofic transitive limit set of an unstable CA? . . .
Summary

Limit sets of a **stable** CA

- ($\Rightarrow$ sofic, mixing)
- SFT
- Nilp.

Limit sets of an **unstable** CA

- Transitive
- Mixing
- Sofic

The End
Summary

Limit sets of a **stable** CA
(⇒ sofic, mixing)
- Kari-XOR

- ECA2, …
- ECA0
- ECA255
- …

Limit sets of an **unstable** CA

- almost all subshifts…
- transitive examples

- ECA122
- language tricks…

- [FK07]

- ECA128

Sofic

Transitive

Mixing
Summary

Limit sets of a **stable CA**

- Kari-XOR
- ECA2, …
- ECA0
- ECA255
- …

(⇒ sofic, mixing)

Limit sets of an **unstable CA**

- ECA122
- [FK07]
- ECA128

**Transitive**

- language tricks…

**Mixing**

- [FK07]

**Sofic**

- almost all subshifts…
- transitive examples

The End