Stability of limit sets of CA

a quick overview and some open problems

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Cellular automata

Definition (Cellular automaton, CA)

A function $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, with

- discrete time,
- discrete space $\mathbb{Z}$ and finite alphabet set A,
- local rule,
- same rule for every cell.

An example...

Elementary CA 54:

$$001, 010, 100, 101 \mapsto 1 \qquad \_ \mapsto 0$$

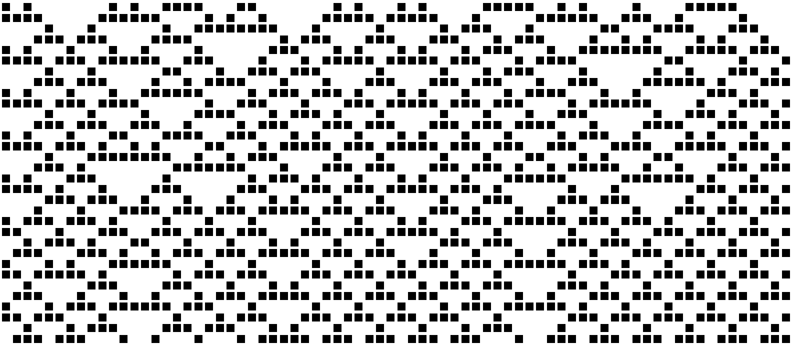

Figure: Space-time diagram of rule 54.

Limit sets

Long-term behaviour of a CA:

Definition (Limit set)

$$\Omega_F = \bigcap_{n \geqslant 0} F^n(A^{\mathbb{Z}})$$

…configurations that can appear arbitrarily far in time.

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Examples

- $\Omega_{\mathsf{Id}} = A^{\mathbb{Z}}$
- ECA 2: $001 \mapsto 1$, $\_ \mapsto 0$
  $\Omega_2 = \Sigma_{\{11,101\}}$
- $\Omega_{128} = \ldots$

Limit set of rule 128

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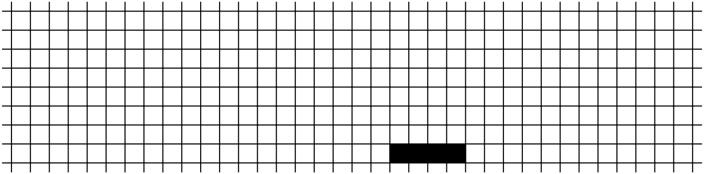

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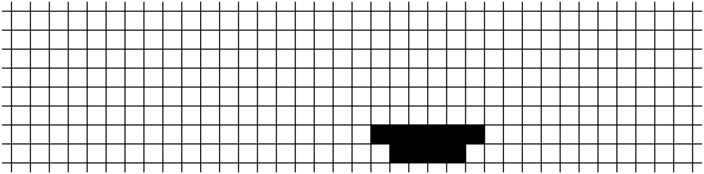

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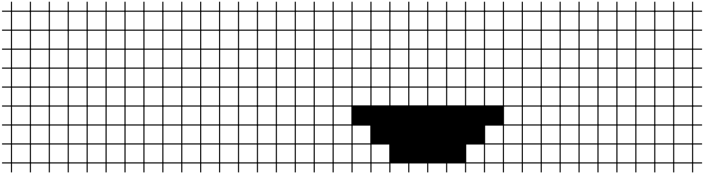

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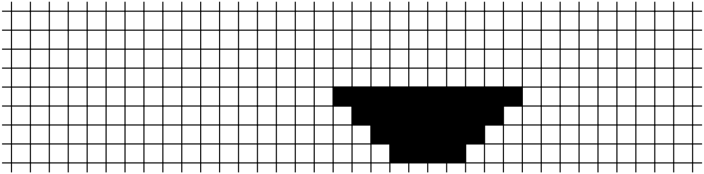

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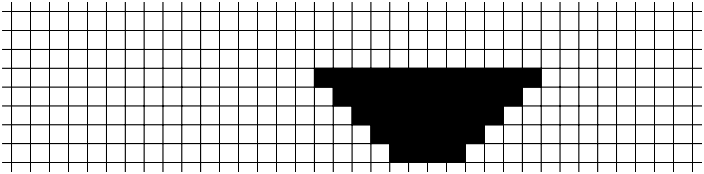

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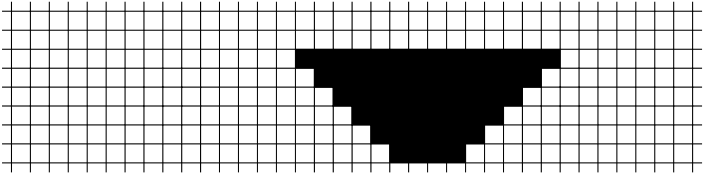

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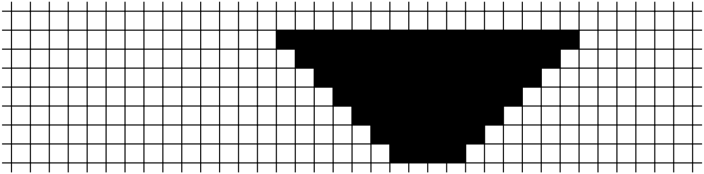

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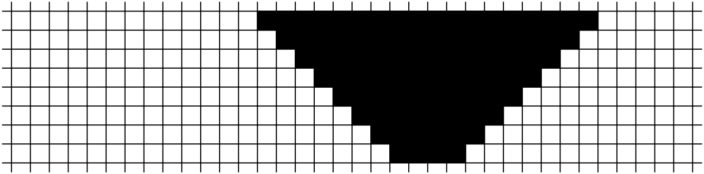

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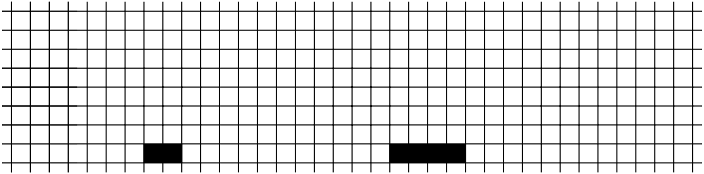

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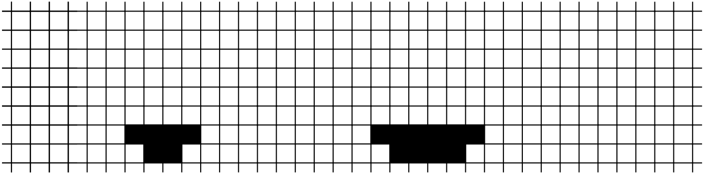

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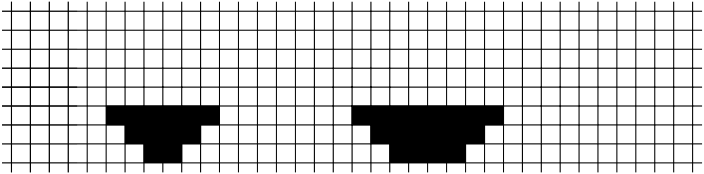

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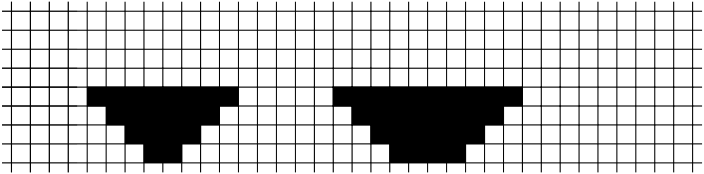

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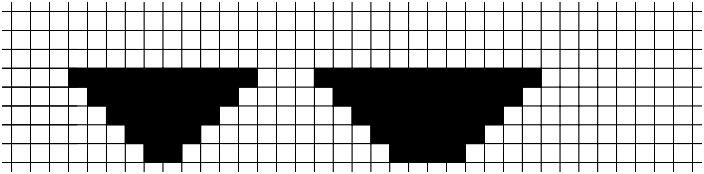

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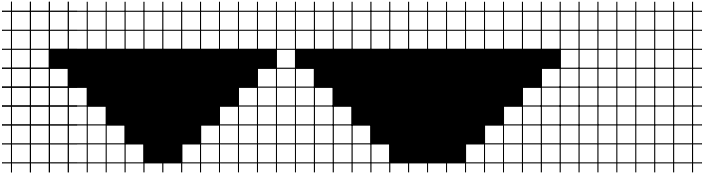

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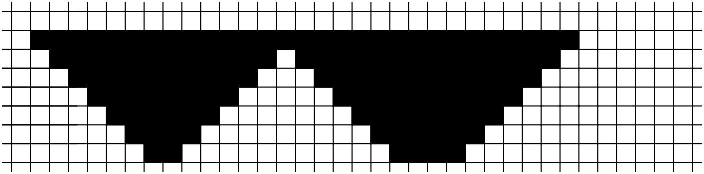

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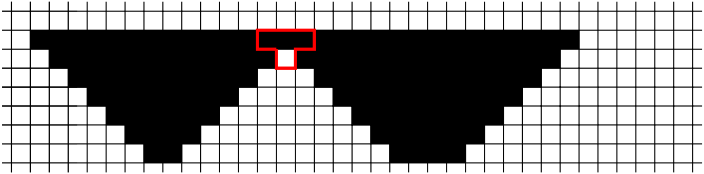

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Stability

$$A^{\mathbb{Z}} \supseteq F(A^{\mathbb{Z}}) \supseteq F^2(A^{\mathbb{Z}}) \supseteq \cdots$$

Definition

F is stable if $\Omega_F = F^n(A^{\mathbb{Z}})$ for some $n \geqslant 0$.
F is unstable otherwise.

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Examples

- The identity is stable...
- ECA 2 is stable: $\Omega_2 = F(A^{\mathbb{Z}}) = \Sigma_{\{11,101\}}$
- ECA 128 is not stable: for all $k \geqslant 0$.
  ($\cdots 0010^{2k}100 \cdots$ is in $F^k(A^{\mathbb{Z}})$ but not in Ω_F)

Definitions

Nilpotent CA: the limit set is a singleton

Language of a subshift: $L(S)$ is the set of words that appear in some configuration of S

SFT: subshift with a finite set of forbidden words

Sofic subshift: factor of SFT ($\Leftrightarrow$ regular language in 1D)

Transitive subshift: $\begin{cases} u \in L(S) \\ v \in L(S) \end{cases} \Rightarrow \exists w \in L(S) : uwv \in L(S)$

Mixing subshift:
$\begin{cases} u \in L(S) \\ v \in L(S) \end{cases} \Rightarrow$ $\exists M \geqslant 0 : \forall m \geqslant M :$
$\exists w \in L(S)$ of size m s.t. $uwv \in L(S)$

Nilpotent $\Rightarrow$ stable

Proposition

F is nilpotent $\quad\Rightarrow\quad$ F is stable

Proof

- Let u be a "universal" configuration.
- Let n such that $F^n(u) = \cdots qqq \cdots$.
- Then $F^n(A^{\mathbb{Z}}) = \{\cdots qqq \cdots\}$, because otherwise there exists a word whose image is not $qq \cdots q$. ⬠

SFT $\Rightarrow$ stable

Proposition

Ω_F is an SFT $\Rightarrow$ F is stable

Proof

- $\Omega_F = \{c$ with a **finite** list of forbidden words $w_1, \dots, w_n\}$
- If $F^k(A^{\mathbb{Z}}) = F^{k+1}(A^{\mathbb{Z}})$ for some k, then F is stable.
- Otherwise, each iteration of F forbids at least one new word.
  ➥ Contradiction because $\{w_1, \dots, w_n\}$ is finite. ⬠

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...is the converse true? ...

A **non-SFT** limit set of a stable CA

Let F be the 1DCA with state set $\{0,1,2\}$, neighborhood $(0,1)$, and local rule

$$f(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) = (0,1) \text{ or } (0,2) \\ 2 & \text{if} \quad (x,y) = (1,0) \text{ or } (2,0) \\ 0 & \text{otherwise} \end{cases} .$$

Proposition

1. F is stable
2. Ω_F is not of finite type

Proof

- $F(\{0,1,2\}^{\mathbb{Z}}) = \{c \text{ s.t. } 10^*1 \notin c \text{ and } 20^*2 \notin c\} = \Omega_F$
- left equality: F forces 1 and 2 to alternate
- right equality: 1 and 2 play the same role ⬠

Necessary condition for stability

We want to characterize the limit sets of stable CA.

Proposition

Every limit set S of a stable CA verifies:

- S is sofic,
- S is mixing,
- S has a uniform config.

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When are these conditions **sufficient**?

Completeness for SFTs

Theorem [Maass]

$$\left\{\begin{array}{l}\text{SFT}\\ \text{sofic, mixing, has a uniform config.}\end{array}\right. \Rightarrow \begin{array}{l}\text{limit set of a}\\ \text{stable CA}\end{array}$$

Even better: If the SFT S verifies the above hypotheses, for **any** factor map $G : S \to S$, we can effectively construct a stable CA F such that $\Omega_F = S$ and $F|_S = G$.

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Can we generalize this?

Completeness for AFTs

Yes, for the class of AFT subshifts [Maass].
AFT: natural weakening of finite type condition...
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Big Question

Can this be extended to sofic subshifts?

➥ Would be a nice characterization of stable limit sets.

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The best answer so far... [di Lena, Margara]

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Some negative results...

- SFT $\Rightarrow$ cannot be unstable
- NM $\Rightarrow$ cannot be unstable [Maass]
- The even subshift $\Sigma_{\{01^{2k+1}0\}}$ cannot be unstable

Transitivity and unstability

All the easy examples of limit sets of unstable CA are not transitive.

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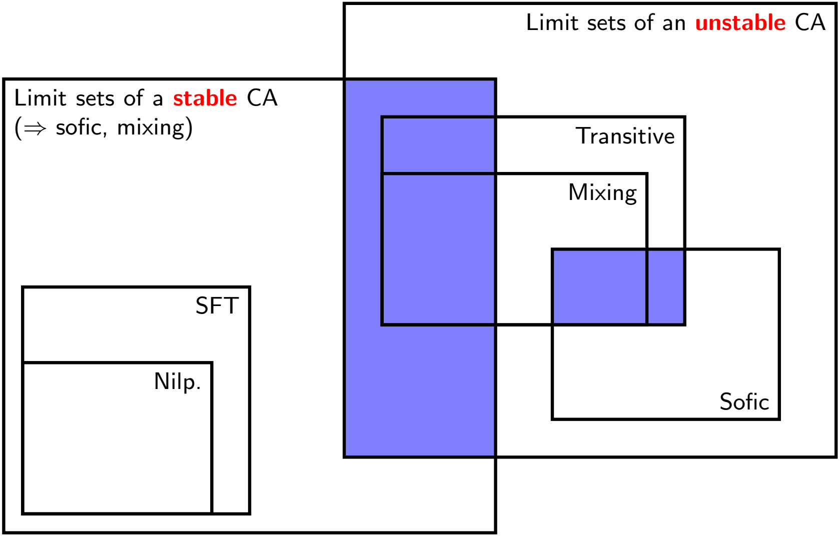
This example is non-sofic. Does there exist a sofic transitive limit set of an unstable CA? . . .

Summary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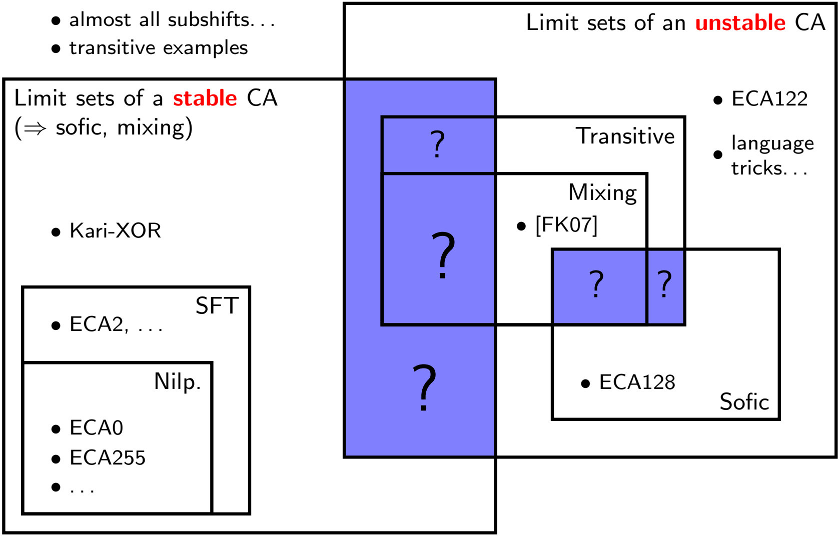
• almost all subshifts...
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Limit sets of an **unstable** CA
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• ECA122
• language tricks...
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• [FK07]
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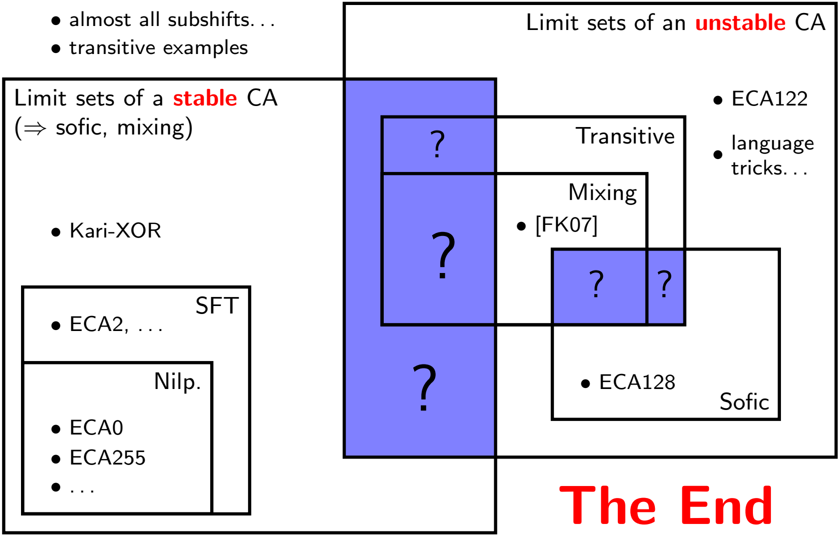

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