Bratteli diagrams, multidimensional substitutions, orbit equivalence

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Bratteli diagrams

A **Bratteli diagram** $\mathcal{B}$ is an infinite graph with:

- vertices by levels $V_0, V_1, V_2, \ldots$ with $|V_0| = 1$
- edges from V_n to V_{n+1}

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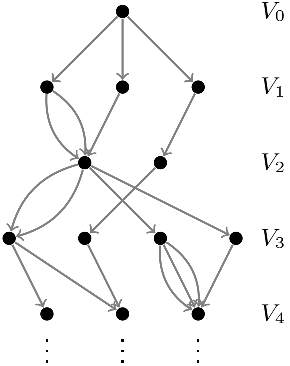

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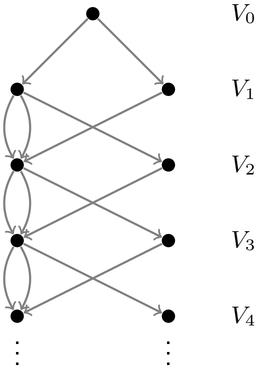
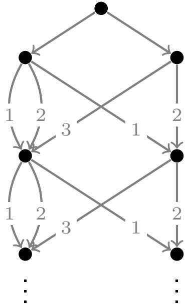


Bratteli diagrams

<u>Ordered</u> Bratteli diagram: order the incoming edges of every vertex



Bratteli diagrams, successor map

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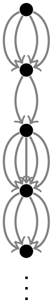
- $X_{\mathcal{B}} = \{$infinite paths starting from $V_0\}$
- **successor** of a path $x \in X_{\mathcal{B}}$:

- let $e =$ first nonmaximal edge of x
- let $f =$ successor of e (given by order on incoming vertices)
- $\mathrm{succ}(x) \;=\;$ minimal path to f $\;+\;$ tail of x from f

Bratteli diagrams

- **Example:** odometers



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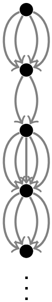

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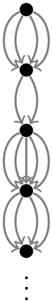

- Systems $(X_{\mathcal{B}}, \mathrm{succ})$ are natural generalizations of odometers
- We now link Bratteli diagrams with substitutions

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- $\sigma : \begin{cases} 1 \mapsto 121 \\ 2 \mapsto 1 \end{cases}$

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- **Subshift** $(X_\sigma, \mathsf{shift})$ where
  - $\mathsf{shift}((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$
  - $X_\sigma = \mathsf{closure}\{\mathsf{shift}^n(u) : n \in \mathbb{Z}\} \subseteq \{1,2\}^{\mathbb{Z}}$
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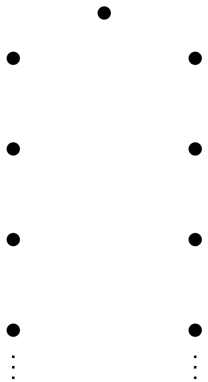
If σ is primitive and aperiodic then (X_σ, shift) is

- minimal
- uniquely ergodic
- has zero entropy
- has no shift-periodic points
- ...

Substitutions and Bratteli diagrams

Bratteli diagram $\mathcal{B}$ associated with $\sigma : \begin{cases} 1 \mapsto 121 \\ 2 \mapsto 1 \end{cases}$

- vertices $V_n = \{1, 2\}$ for all $n \geqslant 1$ (and $|V_0| = 1$)



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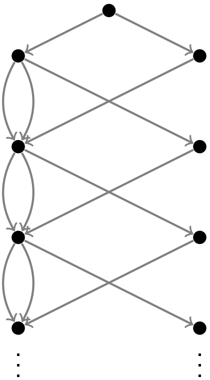

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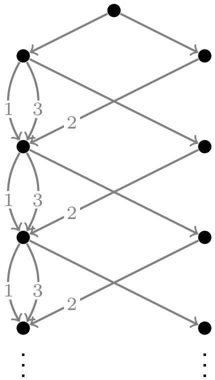

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Topological conjugacy $(X_{\mathcal{B}}, \mathsf{succ}) \cong (X_\sigma, \mathsf{shift})$:

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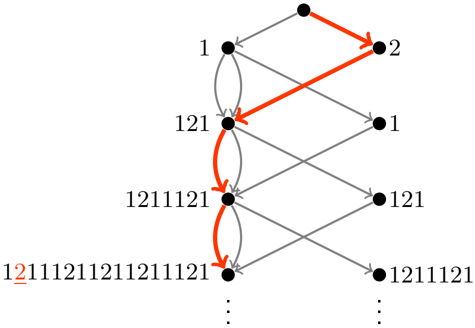

$\cdots$ | 1 | 2 | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 2 | 1 | $\cdots$

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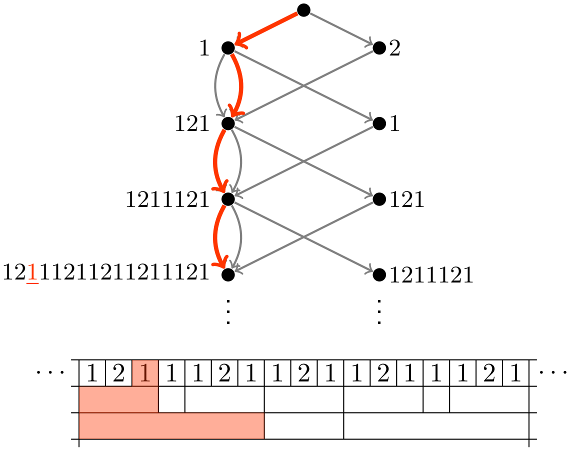

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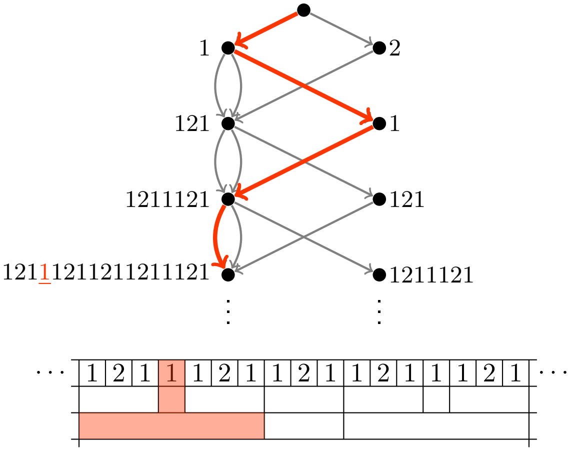

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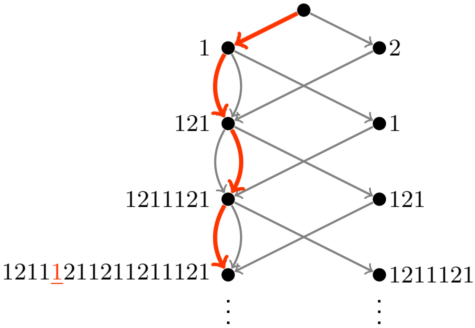

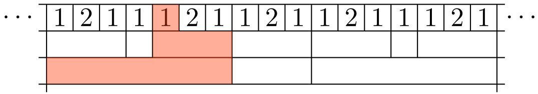

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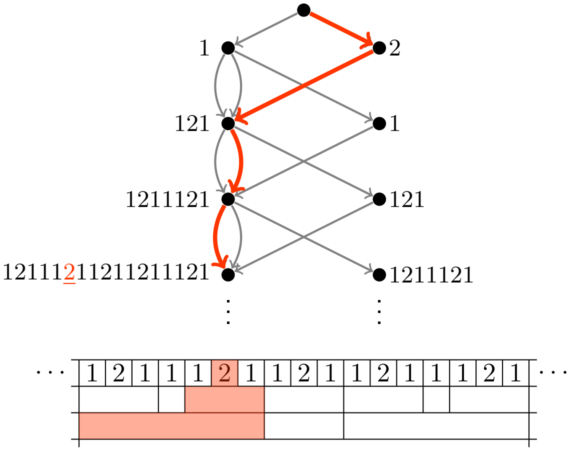

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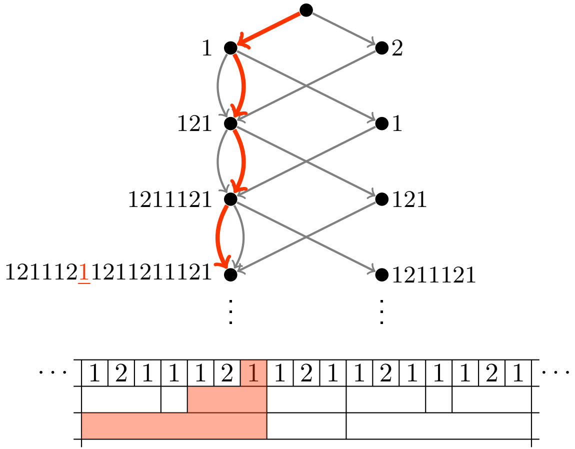

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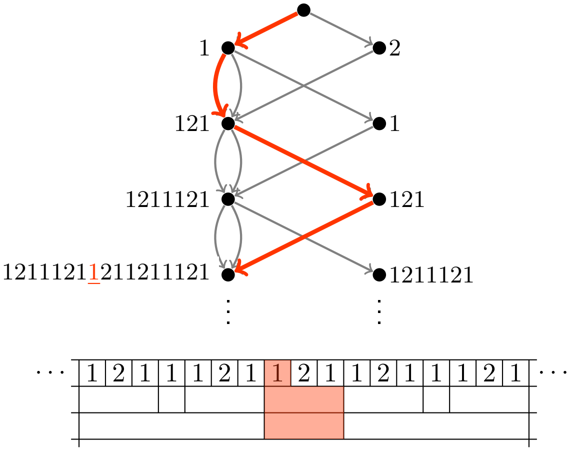
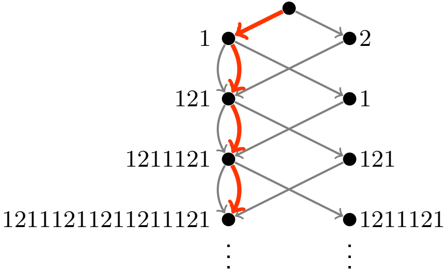


Substitutions and Bratteli diagrams

"Boundary" problem: what is the successor of x_{max}?



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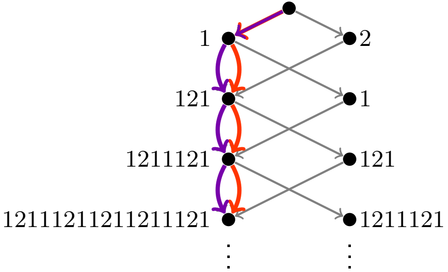

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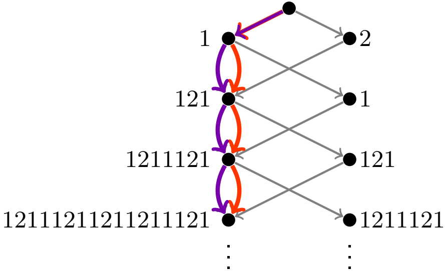

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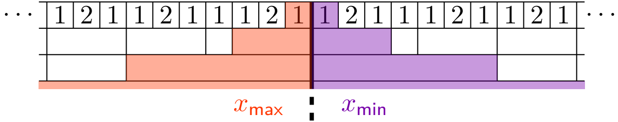

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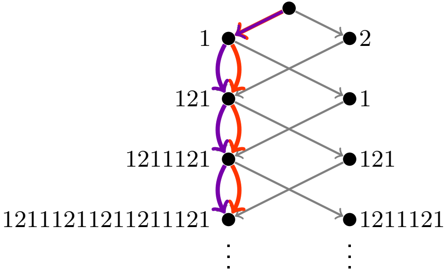

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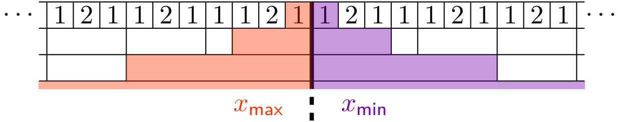

Does this isomorphism hold for every substitution?

Substitutions and Bratteli diagrams

Theorem [Herman-Putnam-Skau 1992]

For every minimal $\mathbb{Z}$-subshift (X, S)
there exists $\mathcal{B}$ such that $(X_{\mathcal{B}}, \mathsf{succ}) \cong (X, S)$.

- Very general result
- Constructive proofs for substitutions [Durand-Host-Skau 1999]

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Original motivation: orbit-equivalence of Cantor minimal systems. . .

Orbit equivalence

Defintion

- Let (X, S) and (Y, T) be two $\mathbb{Z}$-subshifts
- $\mathcal{O}_S(x) = \{S^n(x) : n \in \mathbb{Z}\}$ is the **orbit** of $x \in X$
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More general defintion

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Advantage: actions of **different groups** can be compared

Orbit equivalence of minimal $\mathbb{Z}^d$-actions

What about orbit-equivalence between $\mathbb{Z}^d$-actions on the Cantor space?

Theorem [Giordano-Matui-Putnam-Skau 2010]

For every minimal $\mathbb{Z}^d$-action (X, S) on the Cantor space
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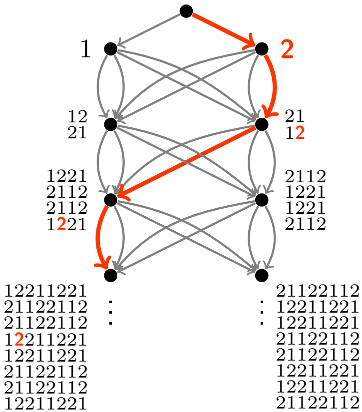
- Surprising: every $\mathbb{Z}^d$-action is realized by an $(X_{\mathcal{B}}, \text{succ})$ (which is essentially a $\mathbb{Z}$-action)
- Does this hold for other groups than $\mathbb{Z}^d$?
  - Not for free groups $\mathbb{F}_k$ [Gaboriau 2000, Gaboriau-Popa 2005]
  - Big open problem for amenable groups

Example $\sigma : 1 \mapsto \begin{smallmatrix}12\\21\end{smallmatrix},\ \ 2 \mapsto \begin{smallmatrix}21\\12\end{smallmatrix}$



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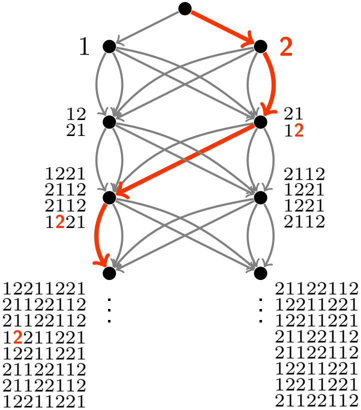

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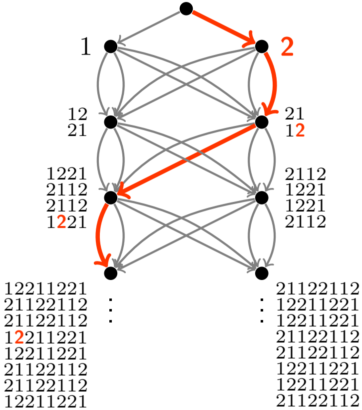

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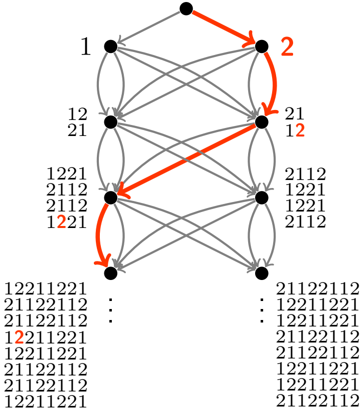

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- **“Boundary” problems** more difficult (“absorption lemmas”)

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- **Question:** which $\mathbb{Z}$-actions are OE to a <u>free</u> $\mathbb{Z}^{d\geqslant 2}$-action?
- One result this direction [Cortez-Petite 2014]:
  for every $\mathbb{Z}$-Toeplitz subshift X and for every $d \geqslant 1$,
  there is a $\mathbb{Z}^d$-Toeplitz subshift which is OE to X

Converse question

Substitutions:

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- understand the "boundary" of the sub-relation induced by Σ on $X_\mathcal{B}$

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- find a suitable 2D subsitution Σ with same underlying $\mathcal{B}$ as σ
- understand the "boundary" of the sub-relation induced by Σ on $X_\mathcal{B}$
- make "gluings" (or use absorption lemmas) to prove that the sub-relation of Σ generates the sub-relation of σ

Converse question

Substitutions:

- Interesting question for some particular X_σ, *e.g.*
  - $\sigma : 1 \mapsto 12, 2 \mapsto 1$
  - $\sigma : 1 \mapsto 12, 2 \mapsto 21$

Possible attack:

- find a suitable 2D subsitution Σ with same underlying $\mathcal{B}$ as σ
- understand the "boundary" of the sub-relation induced by Σ on $X_{\mathcal{B}}$
- make "gluings" (or use absorption lemmas) to prove that the sub-relation of Σ generates the sub-relation of σ
- **Can we hope to do this for every substitution?**

Converse question

2-dimensional Sturmian sequences:

- Let $\mathbf{v} \in \mathbb{R}^3$ and $X \subseteq \{1,2,3\}^{\mathbb{Z}^2}$ be 2D subshift of the coding of the discrete plane normal to $\mathbf{v}$ (2D Sturmian sequence)
- Let $(i_n)_{n \geqslant 1}$ be the Brun-expansion of $\mathbf{v}$
- Express the Brun algorithm with substitutions

$$\sigma_1 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \qquad \sigma_2 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 23 \end{cases} \qquad \sigma_3 : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 13 \end{cases}$$

- Let $Y \subseteq \{1,2,3\}^{\mathbb{Z}}$ be the S-adic subshift generated by $\sigma_{i_1} \cdots \sigma_{i_n}(1)$ for $n \geqslant 1$
- Prove OE between the $\mathbb{Z}^2$-action X and the $\mathbb{Z}$-action Y
  - use 2D dual substitutions Σ_1, Σ_2, Σ_3

Conclusion

- $\mathbb{Z}^d \to \mathbb{Z}$: well understood
- $\mathbb{Z} \to \mathbb{Z}^d$: when is it possible?

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References:

- Fabien Durand, **Chapter 6** of book *Combinatorics, Automata and Number Theory*
- Giordano-Matui-Putnam-Skau, **Orbit equivalence for Cantor minimal Zd -systems**, *Invent. Math.*

Conclusion

- $\mathbb{Z}^d \to \mathbb{Z}$: well understood
- $\mathbb{Z} \to \mathbb{Z}^d$: when is it possible?

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Thank you for your attention