

Bratteli diagrams, multidimensional substitutions, orbit equivalence

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2014-05-09

Bratteli diagrams

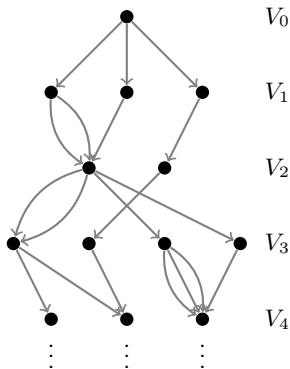
A **Bratteli diagram** \mathcal{B} is an infinite graph with:

- ▶ **vertices** by levels V_0, V_1, V_2, \dots with $|V_0| = 1$
- ▶ **edges** from V_n to V_{n+1}

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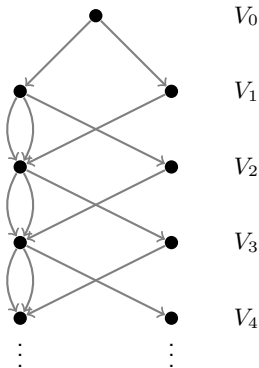
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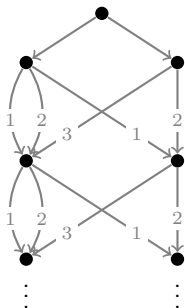
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Bratteli diagrams

Ordered Bratteli diagram: order the incoming edges of every vertex



Bratteli diagrams, successor map

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- ▶ let $e =$ first nonmaximal edge of x
- ▶ let $f =$ successor of e (given by order on incoming vertices)
- ▶ $\text{succ}(x) =$ minimal path to f + tail of x from f

Bratteli diagrams

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- ▶ Systems (X_B, succ) are natural generalizations of odometers
- ▶ We now link Bratteli diagrams with substitutions

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- ▶ $\sigma^\infty(1) = 12111211121112111211121112111211121112111211121121 \dots \in \{1, 2\}^{\mathbb{N}}$
- ▶ **Subshift** (X_σ, shift) where
 - ▶ $\text{shift}((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}$
 - ▶ $X_\sigma = \text{closure}\{\text{shift}^n(u) : n \in \mathbb{Z}\} \subseteq \{1, 2\}^{\mathbb{Z}}$
 - ▶ $u := \sigma^\infty(1) \in \{1, 2\}^{\mathbb{Z}}$

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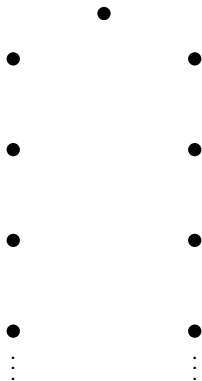
If σ is **primitive** and **aperiodic** then (X_σ, shift) is

- ▶ minimal
- ▶ uniquely ergodic
- ▶ has zero entropy
- ▶ has no shift-periodic points
- ▶ ...

Substitutions and Bratteli diagrams

Bratteli diagram \mathcal{B} associated with $\sigma : \begin{cases} 1 \mapsto 121 \\ 2 \mapsto 1 \end{cases}$

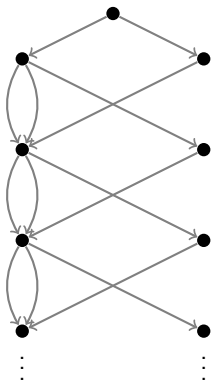
► vertices $V_n = \{1, 2\}$ for all $n \geq 1$ (and $|V_0| = 1$)



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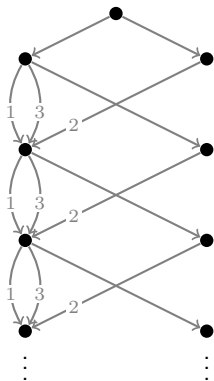
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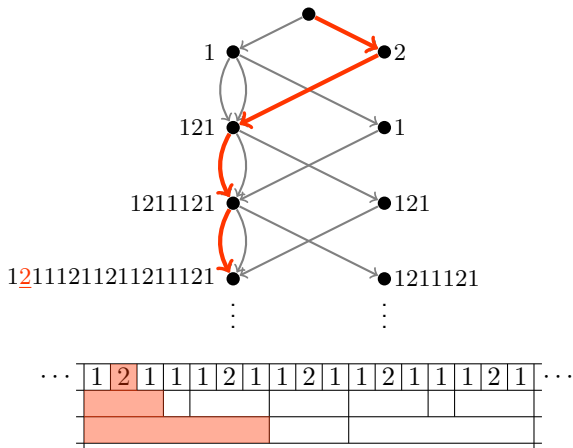
- ▶ vertices $V_n = \{1, 2\}$ for all $n \geq 1$ (and $|V_0| = 1$)
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- ▶ order given by the word $\sigma(f)$



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Topological conjugacy $(X_B, \text{succ}) \cong (X_\sigma, \text{shift})$:

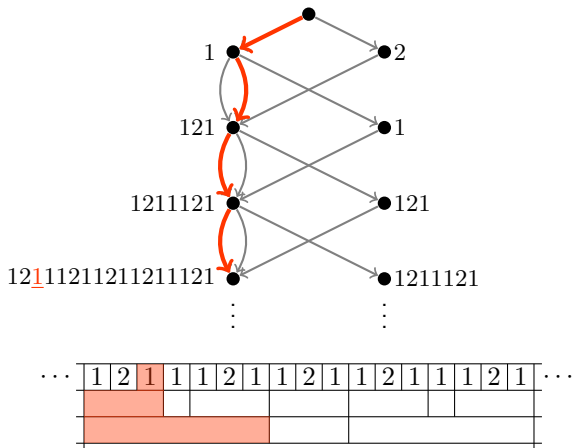
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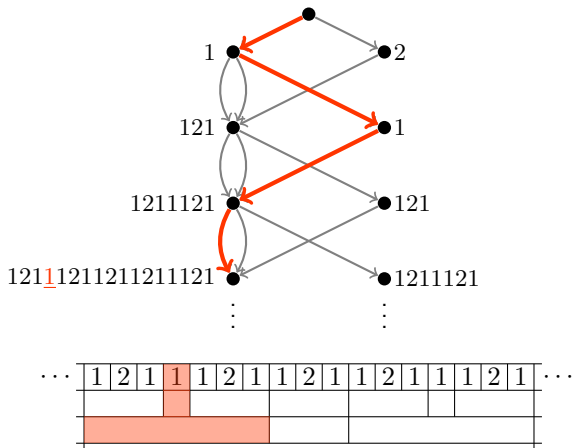
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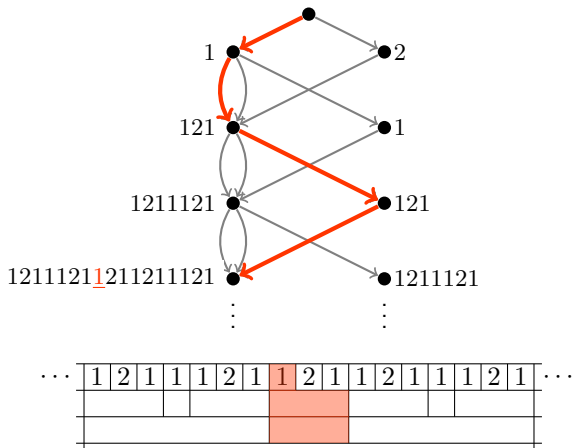
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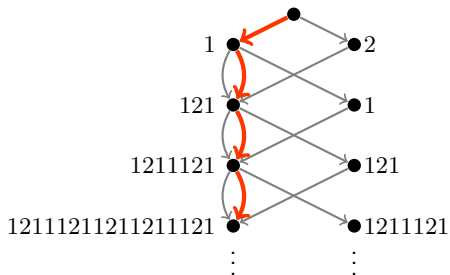
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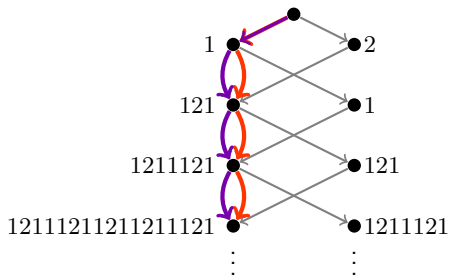
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“Boundary” problem: what is the successor of x_{\max} ?



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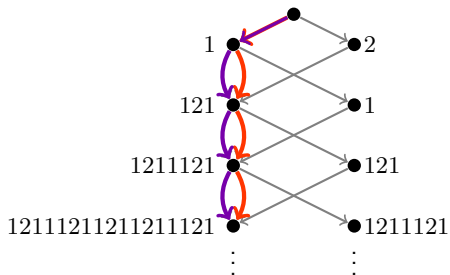
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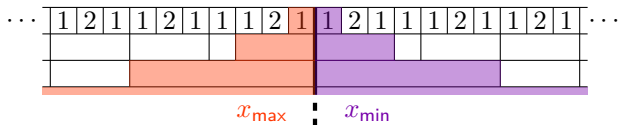
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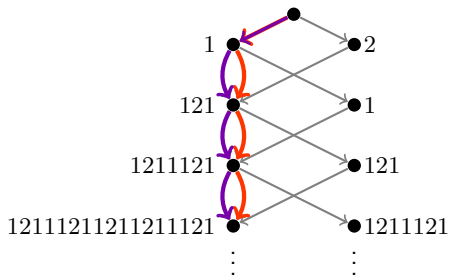


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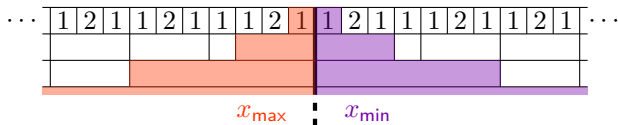


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Does this isomorphism hold for every substitution?

Substitutions and Bratteli diagrams

Theorem [Herman-Putnam-Skau 1992]

For every minimal \mathbb{Z} -subshift (X, S)
there exists \mathcal{B} such that $(X_{\mathcal{B}}, \text{succ}) \cong (X, S)$.

- ▶ Very general result
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Original motivation: orbit-equivalence of Cantor minimal systems. . .

Orbit equivalence

Defintion

- ▶ Let (X, S) and (Y, T) be two \mathbb{Z} -subshifts
- ▶ $\mathcal{O}_S(x) = \{S^n(x) : n \in \mathbb{Z}\}$ is the **orbit** of $x \in X$
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Advantage: actions of **different groups** can be compared

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What about orbit-equivalence between \mathbb{Z}^d -actions on the Cantor space?

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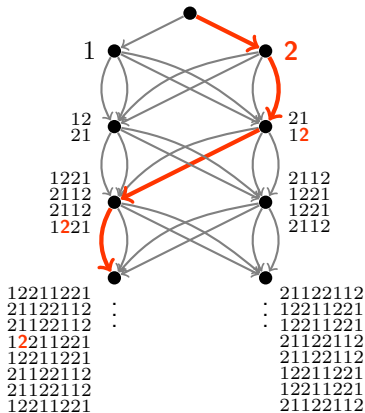
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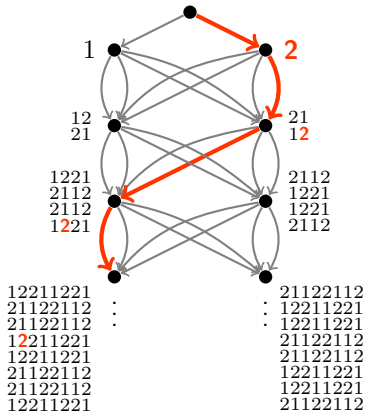
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 - ▶ Not for free groups \mathbb{F}_k [Gaboriau 2000, Gaboriau-Popa 2005]
 - ▶ Big open problem for amenable groups

Example $\sigma : 1 \mapsto \begin{smallmatrix} 12 \\ 21 \end{smallmatrix}, 2 \mapsto \begin{smallmatrix} 21 \\ 12 \end{smallmatrix}$

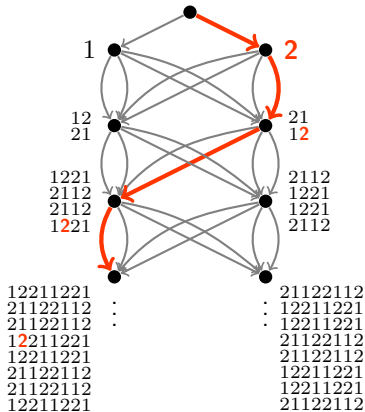


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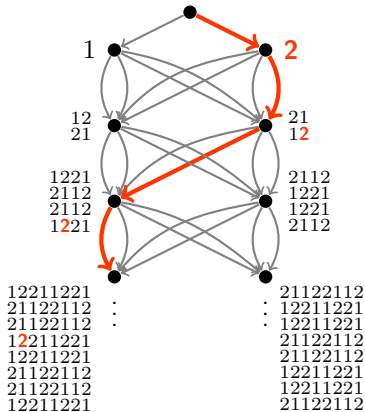
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- ▶ “**Boundary**” problems more difficult (“absorption lemmas”)

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- ▶ **Question:** which \mathbb{Z} -actions are OE to a free $\mathbb{Z}^{d \geq 2}$ -action?
- ▶ One result this direction [Cortez-Petite 2014]:
for every \mathbb{Z} -Toeplitz subshift X and for every $d \geq 1$,
there is a \mathbb{Z}^d -Toeplitz subshift which is OE to X

Converse question

Substitutions:

- ▶ Interesting question for some particular X_σ , e.g.
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- ▶ **Can we hope to do this for every substitution?**

Converse question

2-dimensional Sturmian sequences:

- ▶ Let $\mathbf{v} \in \mathbb{R}^3$ and $X \subseteq \{1, 2, 3\}^{\mathbb{Z}^2}$ be 2D subshift of the coding of the discrete plane normal to \mathbf{v} (2D Sturmian sequence)
- ▶ Let $(i_n)_{n \geq 1}$ be the Brun-expansion of \mathbf{v}
- ▶ Express the Brun algorithm with substitutions

$$\sigma_1 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \quad \sigma_2 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 23 \end{cases} \quad \sigma_3 : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 13 \end{cases}$$

- ▶ Let $Y \subseteq \{1, 2, 3\}^{\mathbb{Z}}$ be the S -adic subshift generated by $\sigma_{i_1} \cdots \sigma_{i_n}(1)$ for $n \geq 1$
- ▶ Prove OE between the \mathbb{Z}^2 -action X and the \mathbb{Z} -action Y
 - ▶ use 2D dual substitutions $\Sigma_1, \Sigma_2, \Sigma_3$

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Thank you for your attention